Chaos in the casino – The math behind card shuffling

Teddy Rocks Maths competition 2025. Produced by Adil Neaz

Introduction

Ever wonder why lady luck never takes your side? Maybe you assumed you must have been a tyrannical dictator in your past life or fell in love with your high school sweetheart too many times while at the casino. But what if I was to let you know the reason you might lose your house is not because of a past life karma or raging hormones, but your lack of understanding of maths. Well, fear not. as if you made it this far, you must be willing to learn the beauty of chaos, probability, and a little prayer here and there. In this essay, you will be exposed to the wonder of chaos theory, card probability's, mathematics behind classic casino shuffles, cracking shuffles mathematically, the distribution of randomness, entropy, and few freaky equations such as the Faro-theorem and the Lyapunov exponent. So, pick up your pants, drop the chips, and pick up some pen and paper as you advance on paying off your 10-year debt to Vegas.

Understand how cards work

. To first begin your revenge plan in humbling a few cocky casinos, you must tackle your first obstacle. Understanding how cards mathematically work. A deck there holds 52 cards, where each is equally crucial in giving you the edge in a poker game and quite helpful in preventing a few loan sharks. You must first understand how a shuffle of the cards deduces the order of randomness within the deck. In turn, you must first be able to deduce the maximum possible ways the cards can be sequenced.

Notation

Let the number of possible card orders be denoted by the variable C.

Let the number of cards in a deck be denoted by the variable N.

Were C = N x (N-1) x (N-2) x (N-3) x … 2 x 1

Therefore, C = N!

. How this is applied in a standard deck, where N = 52:

$$52! = 52 \times 51 \times 50 \times ... \times 1 \approx 8.0658 \times 10^{67}$$

Therefore, C = 52!

To put this into perspective:

- **Un-Ironically greater than the estimated number of atoms in the observable universe (~10^67).**
- **If every person on earth (~ 9 billion) decided their lives were pointless and began to shuffle a deck of cards once per second for the entire age of the universe (~ 13.8 billion years), they still would not exhaust all the possible shuffles.**

. This may seem like an ASTRONMICALLY large number. HOWEVER, don't sign your house away just yet as to how, in practice, a shuffled deck does not explore this space evenly. This will mean that some orders are much more likely to occur than others, depending on the shuffle method. Further, in a real casino game, only a tiny fraction of these possible deck orders is ever realised due to non-random shuffling techniques. In conclusion, you still can tilt the favour of the cards towards your hand IF you can digest the shuffling techniques that are commonly used and further decode them to your advantage.

Common shuffling techniques and how to take advantage

The perfect shuffle – also named the faro shuffle- is a precise way of shuffling cards where the deck is split exactly in half, and the cards are interleaved perfectly (one from each half at a time).

- The flaw of this technique is that it follows a strict mathematical pattern; therefore, it can be deterministic and repeatable and, in turn, can be taken to our advantage.

. 2 types of faro shuffles

Notation

Let N be the total number of cards in the deck.

Let Pi be the position of the card, relative to the total cards in the deck, in its original order (unshuffled), where i is the order of position index ranging from 0 -> N –1.

1. Out-Faro shuffle – the top card (p0) stays on top – example: - let N = 6
- Original deck order: p0, p1, p2, p3, p4, p5
- Split: (p0, p1, p2) and (p3, p4, p5,)
- Shuffled order: p0, p3, p1, p4, p2, p5

2. In-Faro – top card (p1) becomes the second card – example: - Let N = 6
- Original deck order: p0, p1, p2, p3, p4, p5
- Split: (p0, p1, p2) and (p3, p4, p5)
- Shuffled order: p3, p0, p4, p1, p5, p2

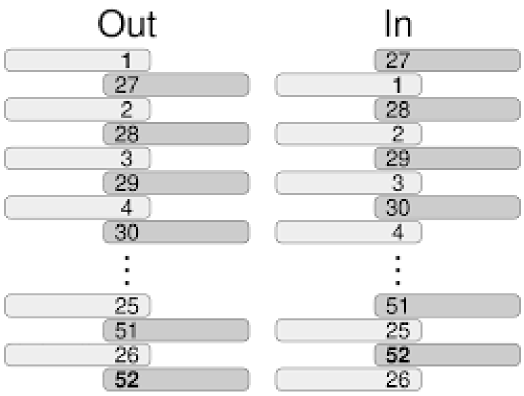
. Mathematical principle – Out-Faro shuffle

- The perfect shuffle can be seen as a permutation (reordering) of the deck.
- For a deck of N cards (where N is even), the new position (f(i)) of card i is:

$$f(i) = \begin{cases} 2i & \text{if } i < N/2, \\ 2i - (N-1) & \text{if } i \geq N/2. \end{cases}$$

In-Faro shuffle – for a deck N cards (where N is even) the new position (f (i)) of card I is:

$$F(i) = \begin{cases} 2i + 1 & \text{if } i < \frac{N}{2} \\ 2(i - \frac{N}{2}) & \text{if } i \geq \frac{N}{2} \end{cases}$$



Returning a sequence to its origin

. How to proof how many shuffles are required before the shuffled sequence returns to the original:

Out-Faro = our principle can be re-written as: $f(i) \equiv 2i \mod (N-1) \quad (\text{for } i \neq 0), \quad f(0) = 0.$

- **Where mod just means when the coefficient is devised by the mod, what is the remainder? – E.g. 7 mod (3) = 1.**
- **The mod will take into account the "wrap around" for values of i > the N/2.**

We seek the smallest K such that: $f^k(i) \equiv i \mod (N-1) \implies 2^k i \equiv i \mod (N-1).$

- **This is after applying k shuffles to our (f(i)) principle and equating efficienets**
- **As this will imply, there is no reminder, meaning the sequence has returned.**
- **This reduces to solving:** $2^k \equiv 1 \mod (N-1).$

. The solution here will depend on the factorization of N – 1

1- **If N – 1 is a prime and 2 is a primitive root, the order to return is N – 2.**
2- **If N – 1 is a composite, the order to return is the least common multiple of the orders modulo prime factors.**

Practiced with a standard deck – N = 52

Out-faro shuffle – our principle can be re-written as: $f(i) \equiv 2i \mod 51 \quad (\text{for } i \neq 0), \quad f(0) = 0.$

- **Applying k shuffles:** $f^k(i) \equiv 2^k i \mod 51.$

We seek the lowest value of k such that: $f^k(i) \equiv i \mod 51 \quad \forall i \neq 0.$

Therefore, equating efficient: $2^k i \equiv i \mod 51 \implies 2^k \equiv 1 \mod 51.$

By factorising 15 = 3 x 17 we can solve: $2^k \equiv 1 \mod 3 \quad \text{and} \quad 2^k \equiv 1 \mod 17.$

The lowest values of k for this to be correct (to equal 1 mod 3 (or 17)) are 2 and 8, respectively. Therefore, the required shuffles must be the LCM of these numbers; therefore, for K = 8.

- **This means that it takes 8 shuffles of the out-faro technique to reset the deck to its origin.**

. In-faro – produced and proved experimentally.

1. **Assign each card a unique number from 0 (top card) to 51 (bottom card) and record the starting order: [0, 1, 2... 51].**
2. **Define the In-faro shuffle by splitting the deck into 2 equal halves and interleave them with the top card (0 originally) now placed under the top card (1 originally).**
3. **Perform shuffles and track the top card – record the position of card 0 each shuffle and stop once the card returns to the top (position 0).**

Expected observations

Shuffle number	Position of card 0	Notes
0	0	Starting position
1	1	Move to position 1
2	3	2(1) + 1 = 3
3	7	2(3) + 1 = 7
4	15	2(7) + 1 = 15
5	31	2(15) + 1 = 31
6	11	2(31) + 1 = 63 mod 52 = 11
…	…	…
52	0	Returns to the top after 52 steps

Therefore, there are 52 steps in an In-faro shuffle for the cards to recycle back to the original hands sequence.

. How this works

1. **The top card follows the sequence: 0, 1, 3, 7, 15, 31, 11, 23 – we can see a general pattern were the position (P) follows: Pn+1 = (2pn) + 1 until the value of Pn+1 > 52 for which then we mod 52 the value and repeat the cycle until mod 52 gives a value of 0 = Pn+1.**

. How can we take advantage of this?

- **If N is known and the order of cards is also known, we can predict what card will be placed when and where.**
- **Due to how this system is deterministic, we can calculate the number of shuffles required for the shuffled sequence to be equal, in order to the original sequence.**

Prediction in a more realistic scenario – Riffle shuffling

- **The Faro shuffle principle relies on the assumption that the casino perfectly conducts the shuffle. However, in reality, it's quite frankly impossible to perfectly split the deck into 2 and place the cards alternatively. This, therefore, induces the probability for error and, in turn, chaos.**

Riffle shuffling – the Faro shuffle in practice where human shuffles are imperfect

. To mathematically describe the shuffle, we need to consider 2 key aspects:

1- **How the deck is split.**
2- **How to 2 halves are interleaved.**

. The first hurdle – splitting the deck. The deck will split into 2 roughly equal halves at a random position.

- **We will use a reasonable probabilistic model that assumes the split (P(k)) follows a binomial distribution:**
- **This shows us the probability for the deck to be split with K amount of cards relative to the total deck N.**

$$P(k) = \binom{N}{k} \left(\frac{1}{2}\right)^N$$

Notation

Let k be the number of cards in the first half, and (NCK) is the number of ways to choose a card from the Deck N.

An ideal riffle shuffle, k is typically around N/2, meaning the deck will always be split ROUGHLY in half (as p (k) will equal 0.5, meaning all permutations are equally possible)

. The second hurdle – interleaving the cards. Due to how we have to take imperfection into account, we have to look at the deck order through a probabilistic perspective to see the likelihood of a card to be placed, relative to the total deck (N) and the cards in the first half (K).

Our model will assume that at each step, the top card from either half is dropped with a probability directly proportional to the number of cards remaining in said half.

Therefore, as the first half has K cards, the second half will have N – K cards.

- **The probability of picking a card from any half at any step is:**

$$P(\text{first half}) = \frac{k}{k + (N - k)} = \frac{k}{N}$$

$$P(\text{second half}) = \frac{N - k}{N}$$

- **The process will then continue until all N cards are interleaved – We can then use the Faro shuffle principle to predict and estimate what card may appear when (respective of whether we use the in or out technique).**

Alternative method to approximate interleaving – The Gilbert-Shannon-Reeds (GSR) model

- **This model is similar to a probabilistic framework that describes how a ruffle shuffle randomizes a deck of N cards.**
- **It provides a precise way to analyse the randomness introduced by a shuffle. This is fundamental in understanding how many shuffles are required to achieve near-perfect randomness.**

. Probabilistic weaving – once the deck is split into 2 parts (denoted by left and right hand) and are interleaved randomly, the GCR model assumes that at each step:

- **You choose the top card from either hand. And that the probability of choosing from a given half is also proportional to the number of cards remaining in that respective half.**
- **Therefore, this model is similar to ours, but now it's relative to a left and right hand, and therefore the probability of picking a card from a half is:**

$$P(\text{next card from left half}) = \frac{L}{L + R}$$

$$P(\text{next card from right half}) = \frac{R}{L + R}$$

How can we use our models?

- **We can now depict how many shuffles it will take before the deck is unpredictable, as the deck tends towards total randomness.**

Measuring randomness – how close a shuffled deck is to fully random permutations (when we can no longer predict with maths☹) by using the "Total variation distance" TVD, which measures:

- **The actual distribution of shuffled decks after a given number of shuffles.**
- **A truly uniform distribution (where all N! ordering are equally likely). We can formulate TVD:**

$$\text{TVD} = \frac{1}{2} \sum_{\text{all orderings}} |P_{\text{shuffled}} - P_{\text{uniform}}|$$

- **A TVD of 1 = deck is completely non-random and, therefore, can be predicted.**
- **A TVD off 0 = deck is perfectly random and unpredictable.**

Application is standard deck were N = 52

- **We are looking for a value of K for where the value of TVD is close to 0 when the total permutations of the deck are equal to 52!**
- **We can then assume that the TVD follows an exponential decay where:**

$$\text{TVD} \approx \frac{2^{52-k}}{52}$$

- **We then solve for a value of K for when TVD is small (approximately 0.05):**

Therefore, it requires 7 riffle shuffles to make a deck indistinguishable from a truly random permutation and, in turn, no longer mathematically predictable.

$$\frac{2^{52-k}}{52} \leq 0.05 \quad 52 - k \leq \log_2(0.05 \times 52) \quad 52 - k \leq \log_2(2.6) \approx 1.38$$

$$k \geq 52 - 1.38 \approx 7$$

Chaos in card shuffling – the flaw of faro

- **Our principle relies on distinct mathematical assumptions in the perfect system, where we assume what we want to happen will, in fact, happen.**
- **However, this leaves a large gap of uncertainty, as no shuffle is truly perfect.**

Chaos theory: sensitivity to initial conditions induces unpredictable outcomes

- **During a shuffle, tiny differences in force, angle, and deck cut lead to vastly different final orders. This is a depiction of a chaotic system.**

Chaotic system – a system where small changes in initial conditions create exponentially growing difference

- **To measure this sensitivity, we use the Lyapunov exponent, which quantifies how fast these tiny errors grow:**

Notation

Let d0 be the number of rising sequences in the initial ordered deck (when d0 = 1).

Let dt be the number of rising sequences at time t (after t shuffles).

Let t be the number of shuffles.

Let λ be the Lyapunov exponent.

Let N (t) be the number of distinct deck orders.

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left(\frac{d_t}{d_0} \right)$$

. If λ > 0: small initial difference will grow exponentially, making the system highly unpredictable. This is a depiction of unpredictable chaos.

. Rising sequences (St) = the subsequence of cards in increasing order (as you go down the cards, how many cards would you have to go through to get to the next value in terms of its positional value? So i.e. many cards in-between 0 and 1 or 2 and 3….).

- **An ordered deck has 1 rising sequence, and after 1 riffle shuffle, the St roughly doubles.**
- **After t shuffles, we assume the number of St is approximately:** $S_t \approx 2^t$

. dt can also be seen as the distance between 2 nearby deck orders. We can assume that this has an exponential growth rate of: $d_t \approx d_0 e^{\lambda t}$

- **As to how in a faro shuffle λ = 0 therefore the number of rising sequences after time t, must be equal to the original order and therefore can be mathematically predictable (assuming original deck order is known).**

Prediction of a 52-card deck model using the faro shuffle:

- **T would be equal to 8 or 52 depending on the respective faro shuffle used, but in both cases, the angle of the log (dt/d0) would now be equal to 1 as after 8 (or 52) shuffles dt=d0 and therefore equalling the log to 0, succumbing the entire equation to be equal to 0**
- **λ = 0: would mean an orderly deterministic system as there is no growth in errors, and therefore the system can still be predicted and any stage**

Chaos in riffle shuffling – here, the chaos built up over time corresponds to how the number of rising sequences in a deck increases with each shuffle, leading to randomness (chaos).

Application with the Lyapunov exponent

- **Remembering that in a riffle shuffle the St, after t shuffles is approximately:** $S_t \approx 2^t$
- **remembering the Lyapunov exponent as well being:** $\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{d_t}{d_0}\right)$
- **If we substitute our values into our equation:**

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{2^t}{1}\right) \qquad = \lim_{t\to\infty} \frac{t\ln(2)}{t} \qquad \lambda \approx \ln(2) \approx 0.69$$

$$= \lim_{t\to\infty} \frac{1}{t} \ln(2^t) \qquad = \ln(2)$$

- **Therefore, in any riffle shuffling, the Lyapunov exponent will be equal to Ln (2). We can now substitute this value into our rising sequence equation, and we arrive at dt = d0*2^t.**
- **Therefore, we can conclude that the small differences between the original deck and that deck after t shuffles will double in magnitude every shuffle.**
- **Intern leading to exponential separation of initially similar card orders, concluding that riffle shuffling must exhibit chaotic behaviour.**
- **However, you may now realize when chaos may be induced within card shuffling and, in turn, be prepared and aware of when it may arise and when you can limit its threat.**

Entropy in shuffling – This is a concept within thermodynamics and information theory that describes the measure of order and randomness of a system.

- **In the context of card shuffling, we can utilize entropy to provide a vital framework for understanding how randomness is introduced into a deck of cards as its shuffled.**

Thermodynamic entropy – entropy can be used to quantify the number of configurations a system can have for a given state.

. Basics of the concepts:

- **High entropy signifies a system with high uncertainty and randomness – a well-shuffled deck will have high entropy due to how there are near-endless possible arrangements (permutations) of the cards.**
- **Low entropy signifies a system with low uncertainty and randomness – an orderly deck would have low entropy due to how it would have only 1 configuration.**
- **Therefore, the act of shuffling will increase entropy, making the cards more unpredictable.**

Shannon entropy in card shuffling – this quantifies the uncertainty of a system in terms of how much information would be required to describe its exact order.

- **This is measured in terms of bits**
- **For a probabilistic distribution over N possible deck orderings, Shannon's entropy, H, is:**

$$H = -\sum_{i=1}^{N} p_i \log_2 p_i$$

Notation

Let H be Shannon's entropy.

Let N be the total possible orderings of a card deck (52!)

Let Pi be the probability of the deck being in the i-th ordering. (CAPS P)

Let i be the position index ranging from 0 -> N –1.

Cases of interest for us

. Perfectly ordered Deck.

- **Only one known state (original order of p0, p1, p2… pn-1).**
- **P1 = 1 for one state, 0 for all others (only one possible order).**
- **So, therefore, H = 0 (no uncertainty and can be predicted).**

. Perfectly shuffled deck

- **Every permutation must be equally likely and, therefore, Pi = 1/52!**
- **Entropy:**

$$H = -\sum_{i=1}^{52!} \frac{1}{52!} \log_2 \left(\frac{1}{52!} \right) = \log_2(52!) \approx 225 \text{ bits}$$

- **This is the maximum possible entropy a system of shuffled cards can have.**

Second law of thermodynamics – this law states that in an isolated system, entropy will never decrease. We can see how this law is proclaimed in card shuffling.

. Natural tendency towards disorder

- **Just like how gas molecules will spread out, shuffling a deck naturally increases its entropy and its unpredictability (like particle positions of a gas)**

. Work required

- **The law implies that energy must be required to reverse the deck order back to its original, therefore, more shuffling must be required to return the deck.**

. Equilibrium in shuffling

- **This law therefore proves that after a set amount of shuffling, the deck must reach a maximum entropy and therefore it cannot become more "random" or unpredictable**

. Our takeaway: we therefore can conclude that a deck cannot infinitely increase in randomness and therefore with enough information, any deck order may be determined.

- **We can also take advantage of the deck during its low-entropy state, where less information is needed and is, therefore, more predictable.**

Conclusion

WELL DONE for how we have successfully taught your feeble mind the beauty of card shuffling. With a little practice, you should be able to easily prove your worth in Vegas with your new fancy math knowledge, as you humble all chaos that comes your way. We have discussed how you may understand and predict card orders and intern positions, mathematically and through rigorous proof. Further, you will now become aware of how to limit the wrath of unpredictable randomness and the latent possibility for inescapable chaos. Including your knowledge of entropy, you may now put your fears to rest as it's theoretically possible to determine any state of a deck at any point (granted you might need a super computer or 2…. BUT THERES HOPE!). Now you shall be confidently able to grant your last kiss to lady luck as you begin to create your own luck through the power of maths.

. Back-up plan if you still don't understand the math: buy a ticket to Dagestan and learn wrestling in the mountains for about 6 years before returning to the Vegas loan sharks.

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