Conic Sections and Orbital Mechanics

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Introduction

In this essay I am going to discuss two areas of mathematics that hold an important role in understanding how the universe works: conic sections (the study of the cone) and some basic orbital mechanics. We will explore how to use these areas to solve a two-body problem giving a satisfying conclusion to the study. This essay aims to be as accessible as possible to people of all aptitudes.

Here is some notation that will be used:

$\therefore$ 'therefore'

$\Rightarrow$ 'implies'

$\propto$ 'is proportional to'

This essay will only concern real numbers.

For any two-body problem concerned in the essay, take the following assumptions:

- All that exists in the universe are two bodies, one large, one small
- Central body is spherical so that its centre of mass is at its geometric centre, and that its mass is evenly distributed
- The larger body influences the small body gravitationally, but the small body does not influence the larger body.

Chapter 1

Conic sections

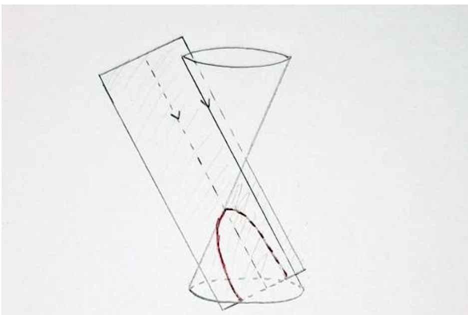
Cones are ostensibly simple shapes that don't pose much of a purpose to mathematicians outside of those strange cone shaped paper cups that exist for some reason. In actuality, when we start dissecting them, cones become an essential tool in understanding how bodies interact with each other gravitationally. By slicing a plane through a cone at different angles we can observe that a certain shape is traced around the outer surface of the cone. The shape formed through this will vary with the angle at which the cone is sliced. To do this, we start by taking a cone and placing an exact copy of it above such that the apex of each cone touch (a bit like a sand timer) as in **Figure 1**. This chapter will discuss the various shapes formed by slicing the cone, and the functions that describe these shapes.

Figure 1

<u>The Parabola</u>

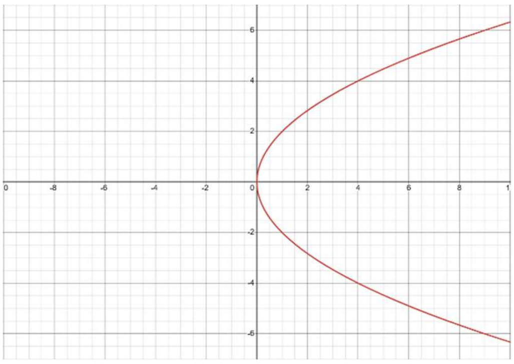
By slicing the cone with the plane parallel to one of the slopes of the cone, a parabola is formed around the outside surface.

Figure 2



We can write the general equation for this parabola as $y^2 = 4ax$, not too dissimilar to a parabola most people are familiar with like $y = ax^2$.

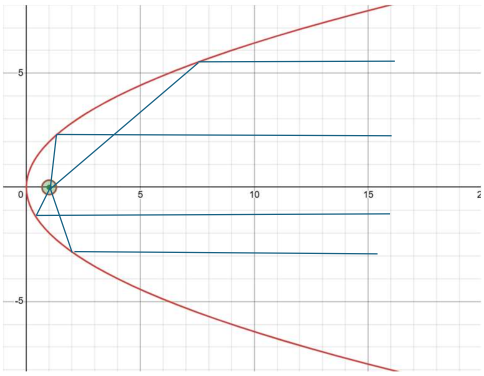
Figure 3 $(a = 3)$



By graphing the function for the parabola in **Figure 3**, we can see a single root at the origin which is the case for any value of a. We will now add some more features to the graph that are not immediately obvious – the *focus* and *directrix*.

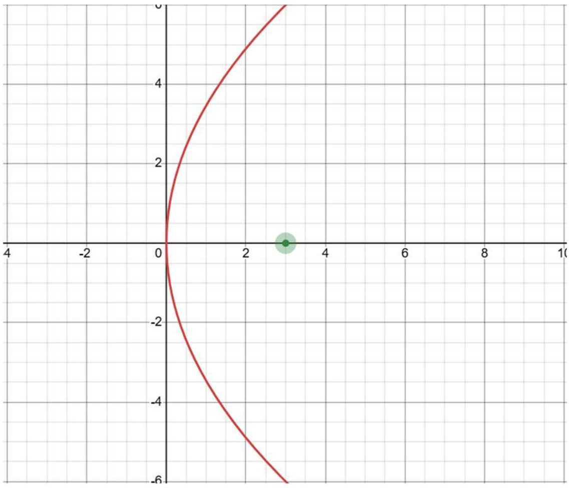
The focus of a parabola can be exemplified by something you may not have realised uses this property: headlights. A car's headlights contain a parabolic reflector which is a special type of mirror with a circular paraboloid shape (a parabola rotated around the x axis). This behaves exactly like a parabola as in **Figure 4**:

Figure 4



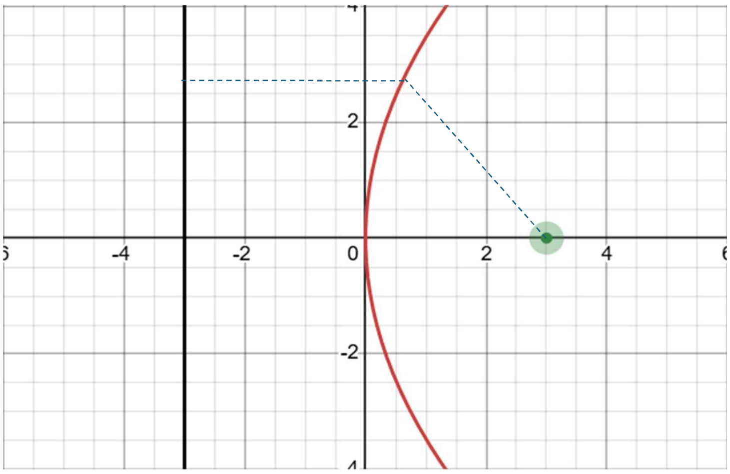
When a light source is placed at the focus of the parabolic reflector, light from it that reaches the reflector's surface will reflect so that it travels parallel to the principal axis ($y = 0$). The focus is the only point where this occurs, the coordinates of which for parabola $y^2 = 4ax$ are $(a, 0)$.

Figure 5 $(a = 3)$



The directrix is a special line whereby $SP = PD$ with S as the focus, D as the directrix and P as the general point on the parabola. The directrix as shown in **Figure 6** has equation $x + a = 0$ (the black line on the left).

Figure 6 $(a = 3)$



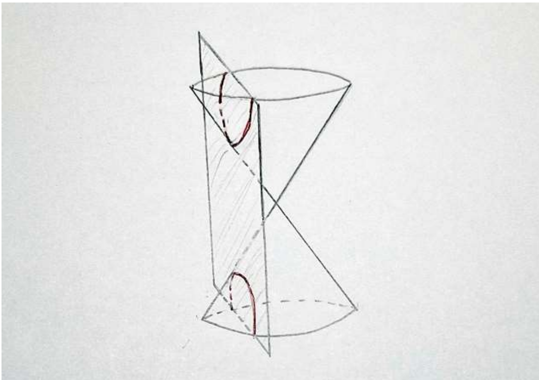
We can use parametric equations to find the general point P. These describe a function with equations so that any value of x and y that satisfy the function are described in terms of t . The parametric equations for a parabola $y^2 = 4ax$ are $x = at^2$ and $y = 2at$ and as such, the general point P is $(at^2, 2at)$. Parametric equations are useful for all types of conic sections, particularly concerning modelling.[1]

[1] Follow this Desmos file to experiment with the parabola, seeing what happens as a varies. https://www.desmos.com/calculator/sonza9uemv

<u>The Hyperbola</u>

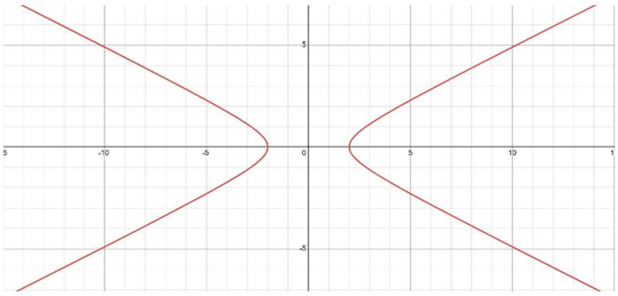
Moving on to the next conic section, when the cone is sliced vertically so that the plane cuts through both the top and bottom cones a hyperbola is formed.

Figure 7



The general equation for a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **Figure 8** shows what this function looks like. It has two asymptotes that are straight lines that pass through the origin.[2]

Figure 8 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



This isn't the only equation for a hyperbola because if the asymptotes meet perpendicular to each other, a special case hyperbola called a rectangular hyperbola occurs. This has a general equation $xy = c^2$.

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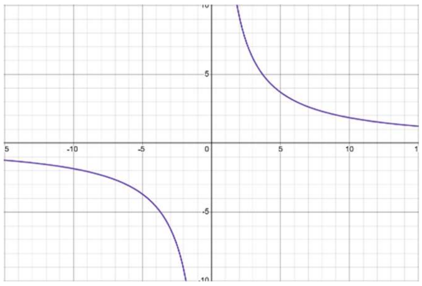
[2] Follow this Desmos file to experiment with the different types of hyperbola. Hyperbola: https://www.desmos.com/calculator/zy7wp0mhy8

Figure 9 $xy = c^2$



To derive the parametric equations for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we use the identity $\cosh^2 t - \sin^2 t = 1$

$$\text{Let } \frac{x}{a} = \cosh t \;,\; \frac{y}{b} = \sinh t$$

$$\therefore\; x = a\cosh t \;,\; y = b\sinh t$$

Or alternatively we can use the identity $\sec^2 t - \tan^2 t = 1$

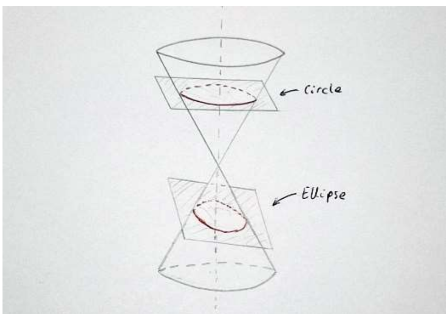
$$\text{Let } \frac{x}{a} = \sec t \;,\; \frac{y}{b} = \tan t$$

$$\therefore\; x = a\sec t \,,\; y = b\tan t$$

<u>The Ellipse</u>

The final shape is the ellipse, formed by slicing through the side of the cone as shown in **Figure 10**.

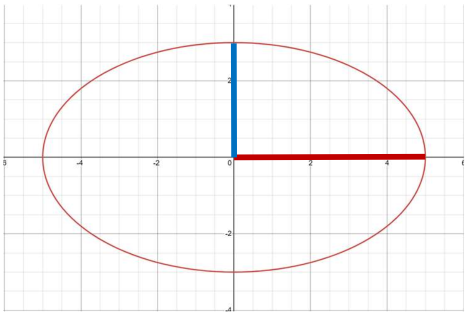
Figure 10



A circle can be made by doing this as well and this is a special case of an ellipse. The function that describes the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

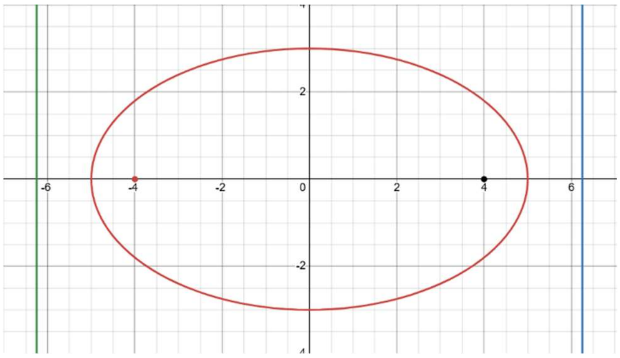
When graphing this function, we see that a and b are axial distances between the centre of the ellipse and its perimeter. In **Figure 11** $a > b$ so a and b are named the semi-major axis and the semi-minor axis respectively.

Figure 11 (a is the horizontal and b is the vertical)



If we now add more features to the graph, we meet some familiar properties: the focus and the directrix. An ellipse has two of each of these – one on either side of its centre[3].

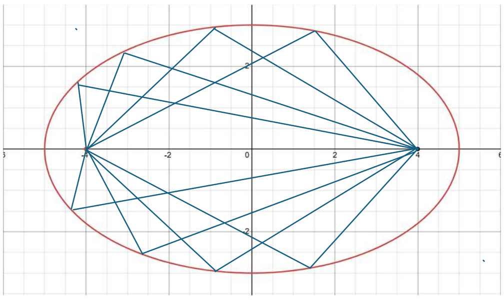
Figure 12



Starting with the foci, we can observe a property in an ellipse that we have already seen in a parabola. Take the previous scenario with the parabolic reflector and try to imagine an elliptical reflector where a light source would be located on one of the foci. The light from the source would all reflect off the elliptical surface and pass through the other focus (example in **Figure 13**).

[3] The plural for focus is foci: an ellipse has two foci.

Figure 13



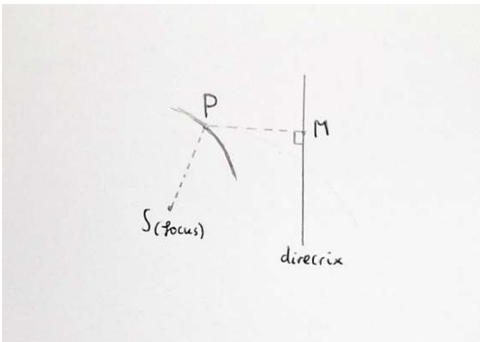
This works for sound waves too, shown by the story of John Quincy Adams in the House of Representatives. Renowned for being able to predict the moves of his political opponents, his desk was positioned on a focus of the elliptical main room of the US capitol building. Subsequently, the sound from the opposite side of the room would reflect off the walls to his desk, such that he could hear even the faintest whisper. There are many buildings around the world that have this acoustic property due to having an elliptical shape.

Figure 14

In order to find the coordinates of the foci mathematically, another property that actually applies to all of the conic sections should be discussed: eccentricity.

<ins>Eccentricity</ins>

Figure 15



In **Figure 15** we can see the distance between the focus and general point P , and from P to the directrix. Here's a quick derivation for P (for an ellipse):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\cos^2 t + \sin^2 t = 1$$

$$\text{Let } \frac{x}{a} = \cos t \ , \ \frac{y}{b} = \cos t$$

$$\therefore \ x = a \cos t \ , \ y = a \cos t$$

$$P(a \cos t \ , \ b \sin t)$$

Eccentricity can be defined as the ratio between the distance from the focus of a conic section to point P on the curve, and the distance between point P and the directrix.

$\therefore \ e = \frac{SP}{PD}$ where S is the focus and D is the point on the directrix such that PD is parallel to the principal axis. e is eccentricity.

In essence, eccentricity is how 'stretched' the ellipse is. As an ellipse becomes more eccentric, its foci become further apart. If $e = 0$, the foci are on the same position forming a circle (hence why it is a special case of ellipse). We can define what type of conic section a function represents by looking at its eccentricity.

If $0 < e < 1$, the conic section is an ellipse.

If $e = 1$, the conic section is a parabola.

If $e > 1$, the conic section is a hyperbola.

The value of e for an ellipse is generalised with $b^2 = a^2(1 - e^2)$ for $a > b$ or $a^2 = b^2(1 - e^2)$ for $b > a$.

With eccentricity, the coordinates of the foci (S) and the equations of the two directrix (M) can be determined.[4]

For $a > b$ $\quad S(ae\,,0)\,,\ S'(-ae\,,0) \quad M\!: \ x = \pm\frac{a}{e}$

For $b > a$ $\quad S(be,0)\,,\ S'(-be,0) \quad M\!: \ x = \pm\frac{b}{e}$

[4] Follow this Desmos file to experiment with the properties of an ellipse: https://www.desmos.com/calculator/uaaeeg3zmz

Area of an Ellipse

The area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is found using integration:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx$$

(Since ellipses are symmetrical horizontally and vertically, I will integrate a quarter and then multiply the result by 4).

Substitute identity $1 - \sin^2\theta = \cos^2\theta$ such that $\frac{x}{a} = sin\theta$

$$\int_0^a b\sqrt{1 - \sin^2\theta}\,dx \Rightarrow \int_0^a b\sqrt{\cos^2\theta}\,dx \Rightarrow \int_0^a b\,cos\theta dx$$

Changing the limits and substitute dx for $d\theta$:

$$x = a\,sin\theta \Rightarrow \frac{dx}{d\theta} = acos\theta \Rightarrow dx = a\,cos\theta d\theta$$

$$x = 0 \rightarrow \theta = 0$$

$$x = a \rightarrow \theta = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} b\,cos\theta \times acos\theta d\theta \Rightarrow \int_0^{\frac{\pi}{2}} ab\cos^2\theta\,d\theta \Rightarrow ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

Substitute identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\frac{ab}{2}\int_0^{\frac{\pi}{2}}(1 + \cos 2\theta)d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{ab}{2}\left(\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - (0)\right)$$

$$= \frac{\pi ab}{4}$$

Since this is a quarter of its area:

Area = πab

This is the formula for the area of any ellipse: π multiplied by the product of the length of the semi-major axis, a, and the semi-minor axis, b. When you think of a circle as a special case of ellipse, the well-known πr^2 is the same as πab, with $a = b$ in this case.

Some may say that the most natural next step when analysing this shape is finding its perimeter, especially considering the similarities between circles and ellipses. For a circle, perimeter (circumference) can be found with $C = 2\pi r$, however, for an ellipse, things are not so simple. There is currently no general formula for the perimeter of an ellipse, mainly because it hasn't been discovered yet. As unbelievable as this sounds (I spent days trying to come up with some solution but nothing I tried worked) mathematicians have found work arounds by using approximations.

However, a big problem that arises when we use these approximations is the effect of eccentricity.

For example, approximation $P \approx \pi(a+b)$ works for ellipses that have an eccentricity close to 0 (almost a circle where $a \approx b$).

When a and b are not approximately equal (larger eccentricity), we have to use two equations to find an estimate for the perimeter:

$P \approx \pi\sqrt{2(a^2+b^2)}$ will give a value larger than the actual value.

$P \approx \pi\left(\frac{3}{2}(a+b)-\sqrt{ab}\right)$ will give a value smaller than the actual value.

Therefore, if we take the mean of the two equations:

$$P \approx \frac{\pi}{2}\left(\sqrt{2(a^2+b^2)}+\left(\frac{3}{2}(a+b)-\sqrt{ab}\right)\right)$$

These approximations are very good, but there are better ones out there like the much more accurate Ramanujan formulas:

$$P \approx \pi\left(3(a+b)-3\sqrt{(3a+b)(a+3b)}\right)$$

$$P \approx \pi(a+b)\left(1+\frac{3h}{10\ \sqrt{4-3h}}\right) \text{ where } h = \frac{(a-b)^2}{(a+b)^2}$$

The perimeter of an ellipse is an enigma that despite the challenge have work arounds for, like these approximations.

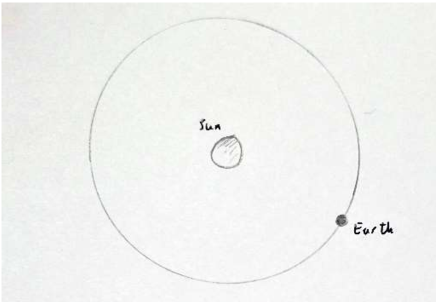
Chapter 2

Orbital motion and gravitation

So far you have seen that conic sections are in some ways very beautiful. We will build on that now moving on to probably the most exciting application of conic sections: orbits. Let's lay out some groundwork first by building a picture.

Take a star, our Sun, and the Earth orbiting around it as in **Figure 17**.

Figure 17



Throughout this chapter we will build on this simple diagram and find out as much as we can about this system using knowledge of ellipses as well as some basic principles of orbital mechanics.

Kepler's laws of planetary motion

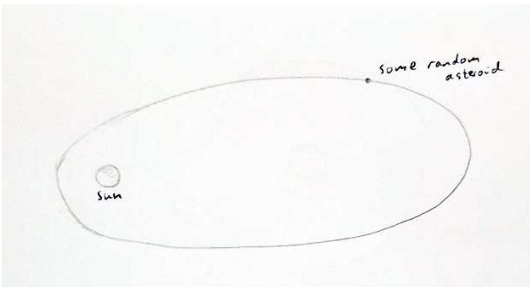
Kepler was an astronomer who lived in the 17th century. He spent a lot of time observing the night sky and through his observations managed to fabricate the laws of planetary motion. Kepler's laws are as follows:

Kepler's First Law:

All planets revolve around the sun in an elliptical orbit with the sun at one of the foci.

Figure 18 shows a highly elliptical orbit in order to illustrate this (note that realistic planetary orbits have an eccentricity close to 0).

Figure 18

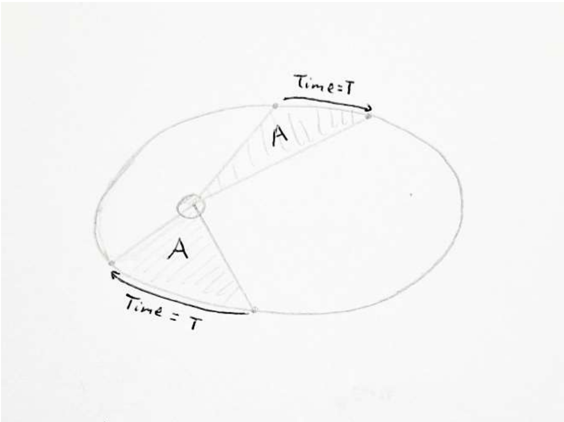


It is immediately clear as to where conic sections are going to come into play from this first law and it is this simple leap in understanding planetary motion that allows us to discover so much.

Kepler's Second Law:

A line joining the planets, and the sun sweeps equal areas in equal time intervals.

Figure 19



The speed of a planet varies throughout an elliptical orbit. In a region where the planet is travelling slower, it will travel less distance in a given time compared to a region where it travels faster. What Kepler's second law states is that the area swept by the planet relative to the star for a particular time period will be equal from any starting point on the ellipse. This seems like a potentially useless piece of information and yet it was this law that led to Newton's discovery of the Universal Law of Gravitation.

Kepler's third and final law of planetary motion is a mathematical relationship:

$T^2 \propto a^3$: *The square of the orbital period of a planet is proportional to the cube of the orbit's semi-major axis.*

This relationship essentially means that the further away the planet is, the longer the time period.

Gravitation

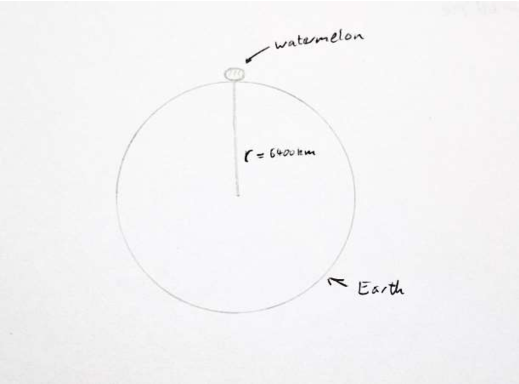
There are two important points to be aware of to discuss gravitation. Firstly, any two masses will attract each other by the force of gravity; two planets; a star and a planet; a star and an apple; two apples. Anything that has mass will exert a gravitational force on another thing that has mass. Secondly, the further away two objects are, the weaker the gravitational force (tying into Kepler's third law).

With this in mind, Newton was able to come up with the equation $F = \frac{Gm_1m_2}{r^2}$ where F is the gravitational force between two objects, $m_1$and m_2 are the respective masses of each object, $G \approx 6.67 \times 10^{-11}\ Nm^2kg^{-2}$ (the gravitational constant of the universe) and r is the distance between the centres of the objects. The equation shows that $F \propto \frac{1}{r^2}$ which follows the inverse square law, indicating that gravitational force weakens with distance. This equation allows for a lot to happen in orbital mechanics including calculating the mass of the Earth.

Mass of the Earth

In **Figure 20** there is a watermelon on Earth's surface and it is distance r away from the Earth's centre (r is equal to the radius of the Earth). The ancient Greeks were able to calculate the radius of the Earth by using some clever trigonometry to do with the sun's rays, hence $r \approx 6400km$.

Figure 20



On earth, the weight of an object can be found by $F = mg$. Comparing this to $F = \frac{Gm_1m_2}{r^2}$, let $m_1 = M$ (mass of the earth), and $m_2 = m$ for the mass of the watermelon.

$$F = \left(\frac{GM}{r^2}\right)m$$

$$\Rightarrow g = \frac{GM}{r^2}$$

Since g (acceleration due to gravity) can be found experimentally by dropping objects and calculating their acceleration, we have an equation solvable for M.

$$g = 9.81ms^{-2},\ G = 6.67 \times 10^{-11}Nm^2kg^{-2},\ r = 6400 \times 10^3m$$

$$9.81 = \frac{(6.67 \times 10^{-11} \times M)}{(6400 \times 10^3)^2}$$

$$\frac{9.81 \times (6400 \times 10^3)^2}{6.67 \times 10^{-11}} = M$$

$$M \approx 6.0 \times 10^{24}kg$$

It is beautiful that we can weigh the earth with mathematics. What is equally fascinating is that we can also find the mass of the sun. To do that we need to talk about centripetal force.

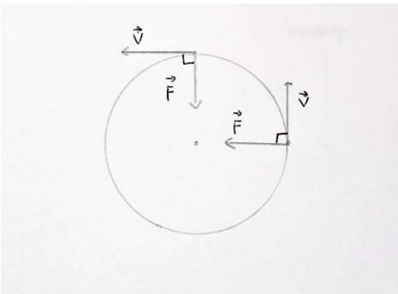
Centripetal Force

When an object travels in a circle around another object at a constant speed, say a car going around a roundabout multiple times because the driver can't figure out which is his exit, that object is accelerating. This is because velocity is a vector and so has direction and magnitude. When travelling in a circle direction constantly changes, therefore velocity also changes (as such $a = \frac{\Delta v}{t}$). Newton's first law of motion states that an object will either remain at rest or travel at a constant velocity if there is no additional force applied to it. An object travelling at a constant velocity moves in a straight line so if an object travels in a circle there must be an additional force that keeps it so. This force is called centripetal force and isn't restricted to one type of force. For a car going around a roundabout, the centripetal force in that scenario would be friction. For a satellite going around Earth, the centripetal force would be the gravitational force between the Earth and the satellite.

One of the equations that used to calculate centripetal force is $F = \frac{mv^2}{r}$, which is derived from Newton's $F = ma$.

Figure 21 shows that for an object travelling in a circle, velocity is tangential to the circle, and the centripetal force acts towards the centre of the circle.

Figure 21



Centripetal forces are not restricted to circles and are also present in ellipses (which makes sense because it is unlikely that the lost driver will go in a perfect circle).

Mass of the Sun

Consider the Earth orbiting the Sun at a constant speed. The Earth's orbit has an eccentricity $e = 0.0167$ so assume that it has a perfectly circular orbit (it almost is a perfect circle, so this is a reasonable assumption). If the Earth is travelling at a constant speed around the Sun there must be a centripetal force, which is in this case the gravitational force between the two. If the centripetal force is equal to the gravitational force, $\frac{Gm_1m_2}{r^2} = \frac{m_1v^2}{r}$ where m_1 is the mass of the

Earth and m_2 is the mass of the sun. Rearranging this we find that $m_2 = \frac{rv^2}{G}$ (notice how the mass of the orbiting body does not matter).

At this point we need to know r in order to find v and thanks to (once again) some neat trigonometry, it is possible to find r . Infact this value is known as the astronomical unit (AU) and is equal to $1.50 \times 10^{11}m$

Since $r = 1.50 \times 10^{11}m$, v can be found with $v = \frac{2\pi r}{T}$ (assuming a circular orbit) where $T = 365\ days = 31536000s$

$$\therefore\ v = \frac{2\pi \times 1.50 \times 10^{11}}{31536000}$$

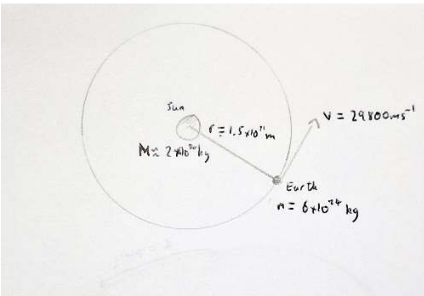
$$\therefore\ v = 29800s^{-1}$$

$$\therefore\ m_2 = \frac{(1.50 \times 10^{11} \times (29800)^2)}{6.67 \times 10^{-11}}$$

$$\therefore\ m_2 = 2.0 \times 10^{30}kg$$

Figure 17 can now be updated with the mass of the Sun and the Earth, the speed at which the Earth orbits the Sun and the radius of the orbit.

Figure 22



This information can be used to find both the centripetal force and gravitational force between the two bodies (they are one in the same in this case) by substituting back into the equations.

This is an orbit with a very low eccentricity to the point where we can assume it is a circle, but things change with more eccentric orbits.

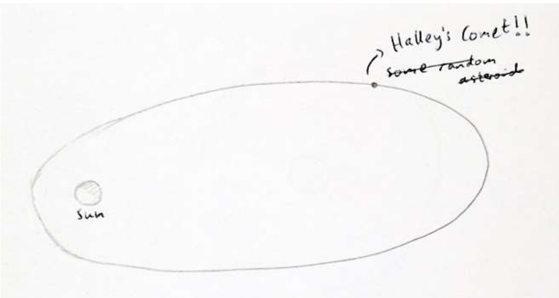
Chapter 3

Application

Halley's Comet

Halley's Comet is one of the few comets that are visible to the naked eye, and that come back round periodically. With a mass of $2.2 \times 10^{14} kg$, it has an orbital period around the Sun of roughly 75 $years$. It has a highly elliptical orbit of $e = 0.96658$ which instead of being close to circular in Earth's case, is closer to being hyperbolic. **Figure 23** shows what the orbit of the comet could look like and here it is clear that the foci of this ellipse are very far apart (the Sun is positioned on one of them), characteristic of a high eccentricity.

Figure 23



In this chapter, we will use everything discussed so far to understand as much as possible about the orbit of Halley's Comet.

$$e = 0.96658$$

$$T = 75\ years$$

$$m_{sun} = 2.0 \times 10^{30} kg$$

$$m_{hc} = 2.2 \times 10^{14} kg \text{ (Mass of Halley's Comet)}$$

With values for e and T we can make a model of the orbital path:

Modelling the orbital path

Kepler's third law $T^2 \propto a^3 \Rightarrow T^2 = ka^3$

k is constant for anything that orbits the sun.

$$\therefore\ T_{Earth}^2 = ka_{Earth}^3$$

$$\therefore (31536000)^2 = k(1.50 \times 10^{11})^3$$

$$k\ =\ 2.97 \times 10^{-19}$$

$$\therefore\ T_{hc}^2 = 2.97 \times 10^{-19} a_{hc}^3$$

$$(75 \times 3153600)^2 = 2.97 \times 10^{-19} a_{hc}^3$$

$$\sqrt[3]{\frac{(75 \times 3153600)^2}{2.97 \times 10^{-19}}} = a_{hc}$$

$$a_{hc} = 5.73 \times 10^{11} m$$

$$b_{hc}^2 = a_{hc}^2(1 - e^2)$$

$$b_{hc}^2 = (5.73 \times 10^{11})^2(1 - 0.96658^2)$$

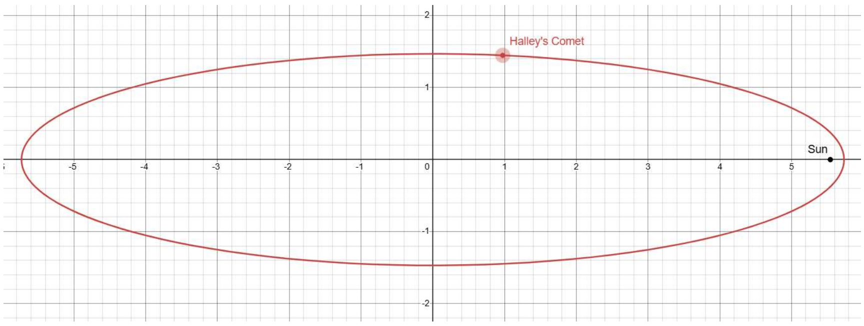
$$b_{hc} = 1.47 \times 10^{11} m$$

With values for a and b, substitute into the equation of an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\frac{x^2}{(5.73 \times 10^{11})^2} + \frac{y^2}{(1.47 \times 10^{11})^2} = 1$$

Displaying this in **Figure 24,** we now have a model for the orbital path of the comet around the Sun. The Sun will be at the right focus $S(ae, 0)\ \Rightarrow\ S(5.54 \times 10^{11}, 0)$

Figure 24



Finding Velocities

General point $P(a\cos t\,, b\sin t)$ allows us to find distance r from the Sun at any point

$$r = \sqrt{(a\cos t + ae)^2 + (b\sin t)^2}$$

At any point the gravitational force $F\ = \frac{Gm_{sun}m_{hc}}{r^2}$

$$F = \frac{6.67 \times 10^{-11} \times 2.0 \times 10^{30} \times 2.2 \times 10^{14}}{r^2}$$

$$F = \frac{2.92 \times 10^{34}}{(a \cos t + ae)^2 + (b \sin t)^2}$$

The centripetal force is equal to the gravitational force (as the comet is in orbit), the magnitude of the velocity of the object at a given point is found by:

$$F = \frac{2.92 \times 10^{34}}{(a \cos t + ae)^2 + (b \sin t)^2} = \frac{m_{hc} v^2}{r}$$

$$\frac{2.92 \times 10^{34}}{(a \cos t + ae)^2 + (b \sin t)^2} = \frac{2.2 \times 10^{14} \times v^2}{\sqrt{(a \cos t + \mathrm{ae})^2 + (b \sin t)^2}}$$

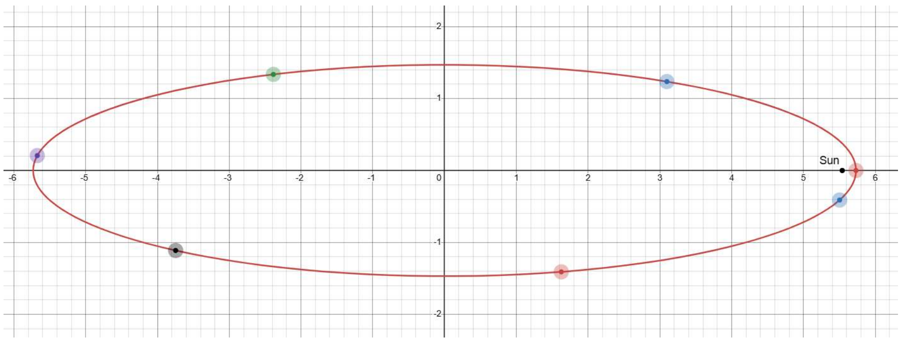
$$v^2 = \frac{1.33 \times 10^{20}}{\sqrt{(a \cos t + ae)^2 + (b \sin t)^2}}$$

$$\mathrm{v} = \sqrt{\frac{1.33 \times 10^{20}}{\sqrt{(\cos t + \mathrm{ae})^2 + (\mathrm{b} \sin t)^2}}}$$

Or generally: $v = \sqrt{\frac{Gm_{sun}}{\mathrm{r}}}$

Through this relationship, Kepler's second law of planetary motion can be demonstrated by taking different points around the orbit and finding the velocity of the comet at each.

Figure 25



(Anti-clockwise around the ellipse starting at the rightmost point):

$$(a \cos 0 , b \sin 0) \Rightarrow v = 83293.0994072 ms^{-1}$$

$$(a \cos 1 , b \sin 1) \Rightarrow v = 22029.7621533 ms^{-1}$$

$$(a \cos 2 , b \sin 2) \Rightarrow v = 12858.8222292 ms^{-1}$$

$$(a \cos 3 , b \sin 3) \Rightarrow v = 10884.9680372 ms^{-1}$$

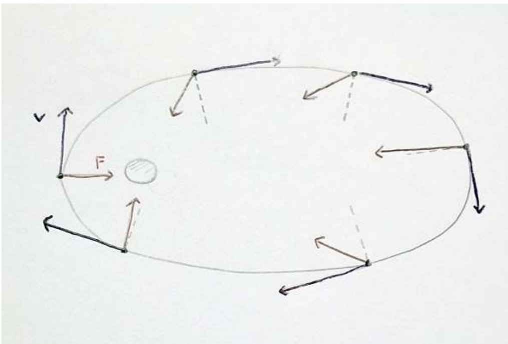
$$(a\cos 4\,, b\sin 4) \;\Rightarrow\; v\; = 11920.073523ms^{-1}$$

$$(a\cos 5\,, b\sin 5) \;\Rightarrow\; v\; = 17873.0403936ms^{-1}$$

$$(a\cos 6\,, b\sin 6) \;\Rightarrow\; v\; = 56779.5073785ms^{-1}$$

The results show that when the distance between the sun and the comet is smaller, the comet travels faster than when it is further away. The point where the comet is at its slowest is at *apoapsis* (point furthest from the Sun) and is at its fastest at *periapsis* (point closest to the Sun). Both of these points lie of the principal axis $y = 0$. This happens because the gravitational force between the two objects changes such that $F \propto \frac{1}{r^2}$ so the orbiting body accelerates throughout the orbit. **Figure 26** shows the direction of the gravitational force and velocity at different points along the orbit.

Figure 26



When the force acts in opposition to the direction of motion, the comet will decelerate as the force reels it back in. When the force acts with the direction of motion, the comet accelerates.

To find the acceleration of the comet at a given point, we could compare $F = m_{hc}a$ to $F = \frac{Gm_{sun}m_{hc}}{r^2}$ so that $a = \frac{Gm_{sun}}{r^2}$ and then substitute in values accordingly.

This shows that $a \propto \frac{1}{r^2} \Rightarrow$ a nonlinear acceleration.

The next step would be to know where the comet is at a certain time in its orbit. To do this we need to employ some different techniques since its acceleration is nonlinear making it hard to write an expression for the position in terms of time.

Finding position for a given time (my method)

Kepler's second law of planetary motion:

A line joining the planets, and the sun sweeps equal areas in equal time intervals.

Using this law, find the area swept in one second starting from periapsis:

$$A = \int_n^a b\sqrt{1-\frac{x^2}{a^2}}\,dx + \frac{b(n-ae)}{2}\sqrt{1-\frac{n^2}{a^2}}$$

To find n a numerical method must be used:

Euler's method: $y_1 = y_0 + h\frac{dy}{dx}$

At $(a, 0)$

$$x_0 = 5.73 \times 10^{11} m$$

$$y_0 = 0m$$

$$vx_0 = 0ms^{-1}$$

$$vy_0 = 83293ms^{-1}$$

$$ax_0 = 0.362ms^{-2}$$

$$ay_0 = 0ms^{-2}$$

$h = 0.001s$ (step length)

Update position and velocity using Euler's method. Update both accelerations by using:

$$a_1 = \frac{GM}{(x_1-ae)^2+y_1^2}$$

$$ax_1 = a_1 \sin\left(\tan^{-1}\frac{(x_1-ae)}{y_1}\right)$$

$$ay_1 = a_1 \cos\left(\tan^{-1}\frac{(x_1-ae)}{y_1}\right)$$

For the first iteration, this gives an estimate for the position at $t = 0.001s$. By putting this method into Python and running it 1000 times, we get a very good approximation for the position at $t = 1s$. The x coordinate of this is our value n to plug into the area equation, allowing us to find A .

Figure 27 (Algorithm to find position at t = 1s)

```
import math

sx0 = 5.7319140918e11
sy0 = 0
vx0 = 0
vy0 = 83293.0994072
ax0 = 0.362169544945
ay0 = 0
h = 0.001
k = 0
while k < 1000:
        sx1 = sx0 + vx0*h
        sy1 = sy0 + vy0*h
        vx1 = vx0 + ax0*h
        vy1 = vy0 + ay0*h
        a1 = (1.334e20)/((sx1-(5.54035352284e11))**2 + sy1**2)
        ax1 = a1*math.sin(math.atan((sx1-(5.54035352284e11)/(sy1))))
        if sy1 > 0:
                ax1 = -ax1
        ay1 = a1*math.cos(math.atan((sx1-(5.54035352284e11)/(sy1))))
        if sx1 > 0:
                ay1 = -ay1

        sx0 = sx1
        sy0 = sy1
        vx0 = vx1
        vy0 = vy1
        ax0 = ax1
        ay0 = ay1
        k = k + 1
print("(",sx0,",",sy0,")") #outputs the coordinates at t = 1
```

With A , we can find the position, m, at 2 seconds by using the following equation:

$$A = \int_m^n b\sqrt{1-\frac{x^2}{a^2}}\,dx + \frac{b(m-ae)}{2}\sqrt{1-\frac{m^2}{a^2}} - \frac{b(n-ae)}{2}\sqrt{1-\frac{n^2}{a^2}}$$

And since A is known,

$$0 = \int_m^n b\sqrt{1-\frac{x^2}{a^2}}\,dx + \frac{b(m-ae)}{2}\sqrt{1-\frac{m^2}{a^2}} - \frac{b(n-ae)}{2}\sqrt{1-\frac{n^2}{a^2}} - A$$

At this point it is very difficult to solve for m so as a workaround, another approximation method will be used, again in python. The equation above states that the difference between A and the calculated area will be 0. Letting the difference be X the equation now looks like this:

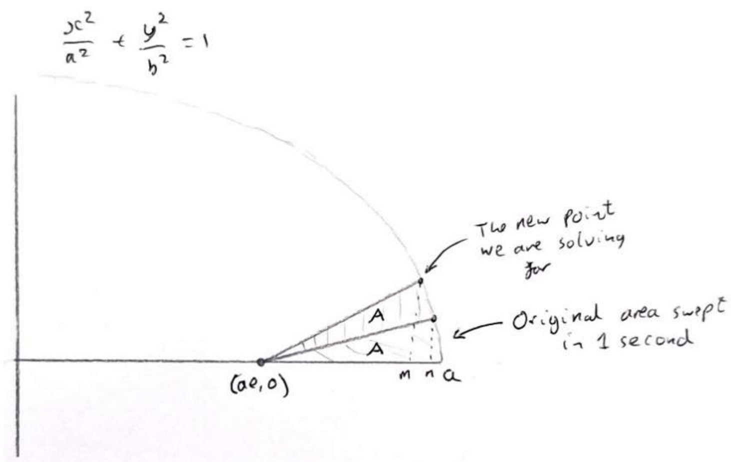
$$X = \int_{m}^{n} b\sqrt{1-\frac{x^2}{a^2}}dx + \frac{b(m-ae)}{2}\sqrt{1-\frac{m^2}{a^2}} - \frac{b(n-ae)}{2}\sqrt{1-\frac{n^2}{a^2}} - A$$

Given that n is known, an estimate for m can be found by substituting in values for m that make X closer to 0.

By making m a bit smaller than n the difference will be greater than 0. By increasing m while keeping $X > 0$, the X will become very close to 0. Eventually, m will be very close to the true value of m and by writing an algorithm that does this process, we can find m to any degree of accuracy.

Once this is done, we can repeat the process for the next second until the desired time is reached with n taking the new value of m and m now being solved for again.

Figure 28



This method will work for any point on the orbit, but the equation used will vary depending on where the point lies relative to the focus. This can be optimised as for larger times the number of calculations is much higher and it will take far longer for the computer run them. An example of this would be to work with a larger area, like finding the area covered in one day and then finding m such that it is one day away from n, the starting point.

Figure 29 (Algorithm to find m between $x = ae$ and $x = a$ for time t)

```python
a = 5.7319140918e11  #semi major axis
b = 1.4694631665e11  #semi minor axis
c = 5.54035352284e11  # a*e (semi major axis * eccentricity)
n = sx0
A = a * b * (math.pi) * 0.25 - a * b * 0.5 * (
    (math.asin(n / a)) + 0.5 * math.sin(2 * math.asin(n / a))) + b * (
        n - c) * 0.5 * (1 - (n**2 / a**2))**0.5  # Area swept in 1 second
print(A)

t = int(input("Enter time in seconds: "))  #enter time travelling for
for i in range(0, t):  #loops the process for number of seconds
        m = round(n, -9)
        num = 9
        while num > -5:  # -5 here can be changed to any integer less than 9 (this allows us to vary our accuracy). -5 gives an answer with 4 decimal places
                Loop = True
                while Loop == True:
                        if m > a:  #necessary as if m > a, X will be undefined
                                m = m - 10**num
                                num = num - 1
                                m = m + 10**num
                        else:
                                Loop = False
                X = 0.5 * a * b * (math.asin(n / a) + 0.5 * math.sin(
                    2 * math.asin(n / a))) - 0.5 * a * b * (math.asin(
                        m / a) + 0.5 * math.sin(2 * math.asin(m / a))) + b * (
                            m - c) * 0.5 * (1 - m**2 / a**2)**0.5 - b * (
                                n - c) * 0.5 * (1 - n**2 / a**2)**0.5 - A
                # X is the difference between the calculated estimate area and the actual known area. The idea is to make X close to 0 by changing m
                if X > 0:  #if X > 0 then the difference can be made smaller so m can be made larger
                        m = m + 10**num
                elif X < 0:
                        m = m - 10**num
                        num = num - 1  #when m is as large as possible for that column, it shifts to the next column down. e.g. hundreds to tens, tens to units.
                        m = m + 10**num
                else:
                        num = -6  #unlikely chance that m gives X = 0
        n = m  #updates n
print("Comet is at (", m, ",", b * (1 - m**2 / a**2)**0.5, ")")
```

With this type of method, the location, velocity and acceleration of the comet at any time along its orbit can be found.

Conclusion

This essay has explored the different conic sections formed from slicing cones in various ways. From these, it focused on the ellipse and used aspects like area, parametric equations and eccentricity in order to create models to describe orbits. Kepler's laws of planetary motion were covered to help find out more about these orbits, and so was Newton's universal law of Gravitation. The equations and content of these laws, paired with knowledge of conic sections was applied to a two-body problem between the Sun and Halley's Comet. In this, with knowledge of the eccentricity of the orbit and using the previously calculated mass of the Earth, an equation for an ellipse that defines the shape of the orbit was found. With this, the parametric equations for this ellipse were utilised to find the distance between the comet and the Sun (which lies on a focus) at any point along the orbit. Then, equations for acceleration and velocity were derived such that with the mass of the comet, the Sun and the distance between the two, both could be found at any point. Finally, in order to find the position of the comet at a given time, numerical methods were employed along with an application of Kepler's second law. Using a python program to run many calculations to ensure it was as accurate as possible, the method results in returning the position of the comet for the given time.

If you are curious about any unanswered questions, or just want to learn more I would recommend checking out the following:

Why is there no equation for the perimeter of an ellipse?

https://youtu.be/5nW3nJhBHL0?si=4lGeIgGTfdKRvoVv

Why slicing a cone gives an ellipse (beautiful proof)

https://youtu.be/pQa_tWZmlGs

Proving Kepler's third law

https://www.sciencefacts.net/keplers-third-law.html

Gravitation by Richard Feynman

https://youtu.be/q_edsSpDzHg

The Two Body Problem (Newton, Kepler) | Fundamentals of Orbital Mechanics 1

https://youtu.be/nJ_f1h49jfM

Desmos Files:

Parabola: https://www.desmos.com/calculator/sonza9uemv

Hyperbola: https://www.desmos.com/calculator/zy7wp0mhy8

Ellipse: https://www.desmos.com/calculator/qtineaf13p

Halley's Comet: https://www.desmos.com/calculator/vaphkr7mmc

Link to Python program: https://replit.com/@19cco01/Comet-position-calculator?v=1