

The Simplicity of Infinite Complexity

Fractals and their wider applications

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Introduction

Fractals exist all around us. You may not know what they are, you may not have ever heard of them, but they are everywhere. Take Broccoli for example. Imagine you have a supersized one, of infinite size. Now if you cut a branch off of this, you would be left with another mini broccoli. Do this infinitely many times, and you would still have another broccoli. It is this self-similarity that all fractals possess, and inherently as a result, infinite complexity as you zoom in further and further into the shape itself.

What actually are Fractals?

“A fractal is a mathematical object that exhibits self-similarity and infinite complexity at different scales.” This means that zooming into a fractal will create an image similar to that of the fractal itself. Additionally, they have infinite detail, and this fundamentally distinguishes them from ordinary geometric shapes. Also, fractals do not have integer dimensions. This property is observed in coastlines, as zooming into individual sections displays ever-complex and increasing perimeters (known as the coastline paradox), and as they are more complex than straight lines, but not completely space-filling shapes (2D); the coastline of Britain has a fractal dimension of roughly 1.25. The Mandelbrot set, Julia sets and the Sierpinski triangle are among the most prominent examples of fractals in mathematics.

Examples in Mathematics

The Sierpinski Triangle

Perhaps the most primitive of Fractals, the Sierpinski triangle discovered in 1915 by Waclaw Sierpinski is created by drawing an equilateral triangle and dividing it into 4 smaller equilateral triangles, then removing the centremost one. Then, for each of the remaining three smaller triangles, the same process occurs, infinitely many times, to create the image on the left. The Sierpinski triangle is a great example of self similarity, as focusing on one of the remaining 3 triangles will always display a new Sierpinski triangle, just with half the side length of the original.

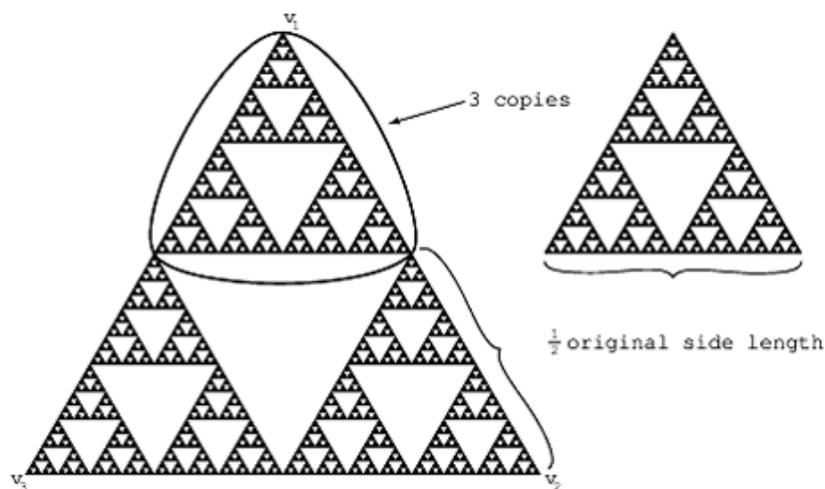


Figure 1: The Sierpinski Triangle

The Mandelbrot and Julia Sets

The Mandelbrot Set and Julia Sets are some of the most studied sets and fractals in all of mathematics. They can be very daunting, especially as they are formed using complex functions, however if picked apart, they are actually a very simple concept.

What are complex numbers?

A complex number, z , is defined as a number in the form $a+bi$, where a is the real part and b is the imaginary part, and i is the square root of -1 . Normally in mathematics, it is written as such: $z = a + bi$, $Re(z) = a$, $Im(z) = b$ and $i^2 = -1$. Complex numbers can also be represented on Argand diagrams, where $Re(z)$ is on the x-axis, and $Im(z)$ is on the y-axis, as seen below.

You may also have noticed that real numbers are just complex numbers where $Im(z) = 0$.

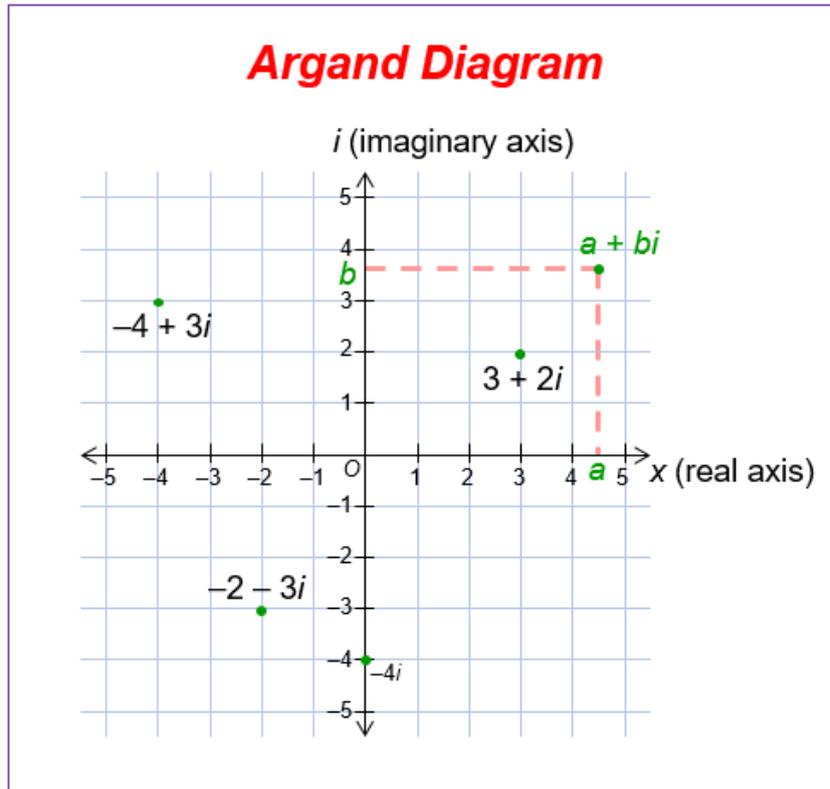


Figure 2: An Argand Diagram

Julia Sets

Julia sets (discovered by Gaston Julia) are defined by the iterative function of complex numbers: $z_{n+1} = z_n^2 + c$. If a point, z , remains bounded under iteration (does not spin off to infinity), it is a member of that specific Julia set. An iterative function is one that is repeatedly applied. For example, the function:

$$x_{n+1} = 2x_n + 1 \text{ would go (when } x_0 = 0\text{): } 1, 3, 7, 15, 31 \dots$$

In the case of Julia sets, we pick a “c” value of any number on the complex plane, for example $2 + 3i$, and there is a unique Julia set for each one.

From there, we pick any value of z_0 on the complex plane, and apply the iterative function, and if it remains bounded after infinite iterations, it is a member of that Julia set. Let’s consider the Julia set of $-1 + 0i$ (i.e -1). Thus, $c = -1$. Take the point $z_0 = 0$. Applying the iterative function ($z_{n+1} = z_n^2 - 1$ as $c = -1$) we get: -1, 0, -1, 0, -1, 0 ...

Clearly, this remains bounded (in fact it is actually called a periodic point of period 2), so is a member of the Julia set of -1.

Further, let’s consider the point $z = \phi$, remembering that $\phi = \frac{1+\sqrt{5}}{2}$. Applying the iterative function ($z_{n+1} = z_n^2 - 1$ as $c = -1$), we get our first term as $\frac{(1+\sqrt{5})^2}{2^2} - 1$, which is $\frac{6+2\sqrt{5}}{4} - 1$, which simplifies to $\frac{2+2\sqrt{5}}{4}$ and subsequently $\frac{1+\sqrt{5}}{2}$. Thus, every time you apply

the function to ϕ , you get ϕ , meaning $z = \phi$ is a fixed point in the Julia Set of -1, and therefore remains bounded and is a member of the set.

So far, all points we have tested have been in the Julia set of -1. Consider $z = -2$. Applying the function we get: $(-2)^2 - 1 = 3, 8, 63, \dots$ which we can see diverges off to infinity. Thus, -2 is not a part of the Julia set for -1.

How can we prove this though? In general, if $|f^n(z)| > R$, where $R = \frac{1+\sqrt{1+4|c|}}{2}$, we can prove for certain that it remains unbounded under iteration, and it not a member of the Julia Set (this can be proven using triangle inequalities and the quadratic formula but will not be covered here).

Using this knowledge, computers can check iterations up to a certain n^{th} iteration, to check if they exceed the R value. If they don't they, can colour it in black, and if it does, they colour it in white. Computers can also be use different colours to display how fast the points iterate off to infinity (ie. pass R), creating pictures such as these:

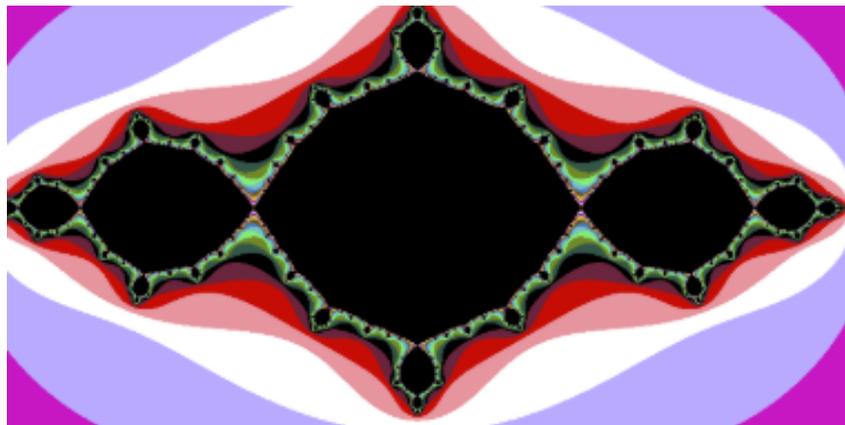


Figure 3: Julia Set of -1

The Mandelbrot Set

The Mandelbrot Set was discovered in 1980 by Benoit Mandelbrot. It is similar to Julia Sets whereby it is created by the iterative function $z_{n+1} = z_n^2 + c$, however instead of being a set of values of z , it is a set of values of c . Therefore, there is only one Mandelbrot set, unlike the many Julia sets. In simple terms, instead of choosing a point on an Argand diagram to be your z value, as previously, you choose a c value, and using $z_0 = 0$, we can test whether that value of c remains bounded under iteration. If it does, then that value is in the set, otherwise, it is not.

For example, take $c = -1 + 0.25i$.

Applying the iterative function with $z_0 = 0$, we get $0^2 - 1 + 0.25i = -1 + 0.25i$. Continuing on, have the sequence: $0, -1 + 0.25i, -0.0625 - 0.25i, -1.058\dots + 0.28125i, \dots$

Iterating infinitely, this remains bounded, thus $-1+0.25i$ is part of the Mandelbrot set.

For another example, take $c = 1$.

Applying the function we get: $0, 1, 2, 5, 26, \dots$

Clearly, this remains unbounded under infinite iteration, and thus is not part of the Mandelbrot set.

Properties of the Mandelbrot Set: The Mandelbrot set is a connected set, and is contained in a disk of radius 2 centred at zero.

The Julia sets of the points in the Mandelbrot set are always connected. This is because points in the set are “stable”, and thus are able to keep the Julia set together. Points outside the set are chaotic and unstable, and thus cause their Julia set to break apart.

Zooming into pre-periodic points (ie. points that are periodic [repeating] but do not return to their original z_0 value) gives a self-similar image of the Mandelbrot set, meaning it looks the exact same as zooming into another pre-periodic point.

Representation: The Mandelbrot Set also has a few methods of visualising. The most common algorithm used is the “escape time algorithm”, just like with Julia Sets. When a point c is not part of the mandelbrot set, the number of iterations before $|z_n|$ exceeds 2 determines the escape time. This is because 2 is in the basin of attraction (values inside basin of attraction diverge off to infinity), and thus no members of the Mandelbrot set have $|z| > 2$.

Colours can then be used to plot the different escape times, sometimes with smooth gradients and other times with block colours. The more iterations the computer performs, the more accurate the image will be, however the number of iterations increases computing power required drastically. This creates images such as this:

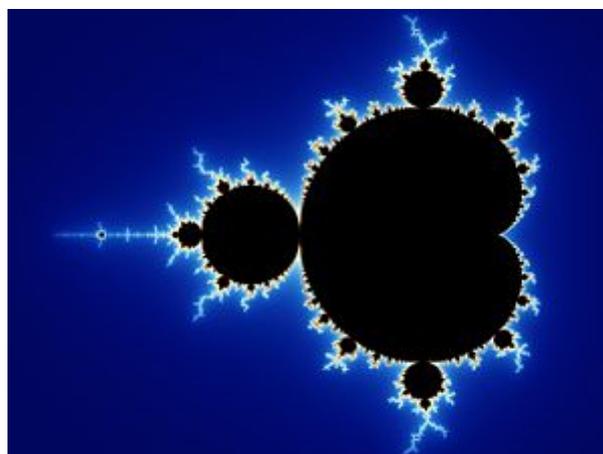


Figure 4: The Mandelbrot Set (Black)

Applications - Why do we care?

Fractals are still studied to this day in Mathematics, so much so that universities have entire modules dedicated to the studies of these geometrically complex shapes. However, for any non-mathematicians fractals may seem useless. In reality, there are many practical applications of Fractals.

Realistic Terrain Generation

Fractals exist in the landscape all around us, and this is seen as part of North America looks exactly like parts of Russia, and parts of Australia look like parts of Africa.

Therefore fractals can be used in terrain generation to replicate this.

For example in Minecraft, fractals are used to create natural landscapes and smooth, flowing terrain, eliminating the need for large data sets. They use multiple fractal variables to create the self-similarity, while also creating variation (i.e variables for max height, min height, variation in terrain shape, location of structures ...)

CGI also uses this function for production in hollywood films.

This is because the iteration of complex geometric shapes helps to create intricate designs intended for high quality realism.

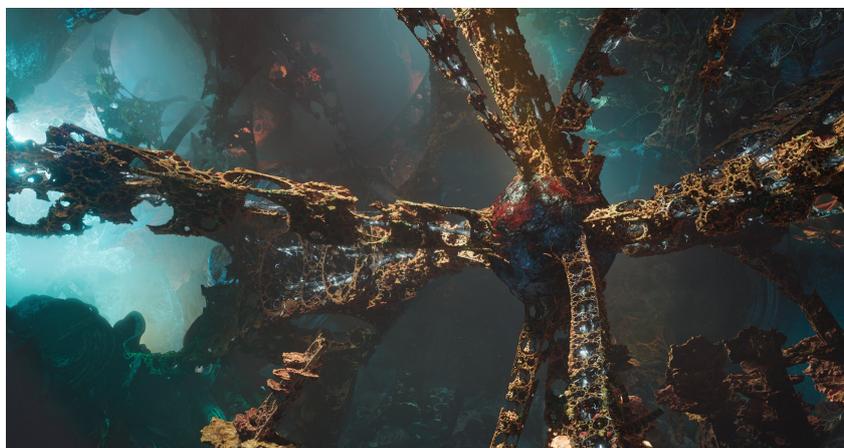


Figure 5: Guardians of the Galaxy Vol.2 CGI

Economics

In economics, fractals are used for fractal analysis of market trends. This works because the system can detect small fluctuations in the market which often resemble larger fluctuations.

This is useful for stock analysis as fractals can notify when the stock market is likely to plummet or rise, which is useful for stockbrokers and people trying to profit off their investments.

This is actually how Benoit Mandelbrot initially discovered this property of fractals. As an economist, he discovered that stock fluctuations followed a repeatable pattern, and he went on to write about his find called the Misbehaviour of Markets.

Astronomy and Cosmology

Interestingly, fractals can also be used to model the universe.

This helps us to understand how galaxies cluster together on a larger scale.

This is because the spread of galaxies is not uniform but clustered in a structure which resembles complex fractal geometry. With this observation, we can better understand the probable properties of the larger picture of the universe. This known as the cosmic web:

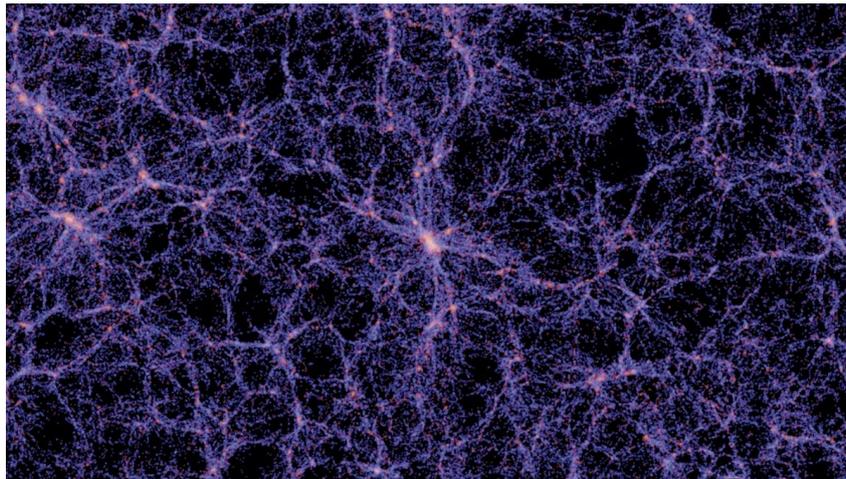


Figure 6: The Cosmic Web

Conclusion

Fractals are infinitely complex shapes, with a wide range of applications, but I hope you now have take away that there is method to mathematical madness, calm in chaos and simplicity in infinite complexity.