Chasing Shadows: Odd Perfect Numbers

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Introduction

Imagine you lost your car keys. How long would you keep looking for them? An hour? A day? Maybe a week if you were *really* desperate. Now, imagine searching for something for over 2,500 years and still having no idea whether it exists. That's exactly what mathematicians have been doing with odd perfect numbers. The ancient Greeks initially studied these, believing they had mystical properties, and started searching for an odd one over two millennia ago. If you'd lost something 2,500 years ago, you'd probably have given up by now – but mathematicians are nothing if not persistent. So, what exactly are perfect numbers?

Throughout history, perfect numbers have had some spiritual significance, and their mathematical perfection mirrored the order of the divine. The ancient Greeks were the first who recognised their mystical qualities, as early as Pythagoreans in 525 BCE, but it was the Christian scholars who wove them into theological thought. This idea was continued by Nicomachus of Gerasa, a philosopher in the first century CE, but most famously supported by St. Augustine. Consider the number 6, and the number 28, the two smallest perfect numbers. Two significant numbers in the Christian religion. 6 reflects the six days of creation in the Book of Genesis, and 28 represents the 28-day lunar cycle, implying sacrality and that these numbers are examples of "Heavenly". St Augustine wrote in "The City of God", "Six is a number perfect in itself, and not because God created all things in six days; rather, the converse is true. God created all things in six days because the number is perfect." This view reveals a deeper truth, which is that perfect numbers were not coincidental to biblical events but are seen as a foundation to God's design. They allow mathematicians to ponder their significance in the beautiful language of mathematics.[1]

A perfect number is a positive integer that is equal to the sum of its proper divisors, excluding itself. The first is 6, since its divisors are 1,2 and 3, and 1+2+3=6, and the next is 28, for whose divisors are 1, 2,4,7 and 14, and 1+2+4+7+14 = 28. Currently, there are 51 known perfect numbers, all of which are even. Thus, it's clear to see that *even* perfect numbers exist, but the real mystery is whether an odd perfect number exists.

Perfect Numbers:

Let's start with the basic prerequisites before diving into perfect numbers:

An arithmetical function is multiplicative if $f(mn) = f(m) \times f(n)$ when m and n are coprime (their highest common factor is 1), and totally multiplicative if this holds for any values of m and n.

[1] I'd like to acknowledge Dr Charafi, my mathematics teacher for introducing me to the beauty of mathematics, and how we should treat mathematics as the language of which God has written the universe. He told me that should I understand the language of mathematics fluently; I would be able to read the world around me in ways that others cannot.

Let's call f(n) a multiplicative function and say $F(n) = \sum_{d|n} f(d)$. Thus, we can say that F(n) is multiplicative too.

Proof:
Let m, n be positive integers with gcd[2](m, n) = 1 and f be a multiplicative function.

Let $$F(n) = \sum_{d|n} f(d).$$
Therefore,
$$F(mn) = \sum_{d|mn} f(d).$$
For each d there must exist some positive integers r, s such that $rs = d$ and $r \mid m$, $s \mid n$. Therefore,
$$\sum_{d|mn} f(d) = \sum_{r|m, s|n} f(rs) .$$

It follows from gcd(m, n) = 1 that gcd(r, s) = 1 and since f is a multiplicative function $f(rs) = f(r)f(s)$.

Therefore,

$$\sum_{r|m, s|n} f(rs) = \sum_{r|m, s|n} f(r)f(s) = \sum_{r|m} f(r) \sum_{s|n} f(s) = F(m)F(n) .$$

Thus, $F(mn) = F(m)F(n)$, given that gcd(m, n) = 1 and so $F(n)$ is a multiplicative function.

Now, it's important to define $\sigma(n)$ as the sum of all unique positive factors of the number n. Therefore, n is perfect if and only if $\sigma(n) = 2n$ (for $\sigma(n)$ includes the factor of n itself). Thus, it now follows that $\sigma(x)$ is multiplicative as let $f(x) = x$. $f(x)$ is clearly multiplicative as $f(mn) = mn$ and $f(m)f(n) = m \times n = mn$ so $f(mn) = f(m)f(n)$.

$$\sigma(x) = \sum_{d|x} f(d) \text{ where } f(d) = d, \text{ which we have shown to be multiplicative.}$$

Therefore, from our proof, $\sigma(x)$ is multiplicative and $\sigma(mn) = \sigma(m)\sigma(n)$, for gcd($m, n$) = 1.

Even Perfect Numbers:

The first progress made on perfect numbers was by Euclid in the 3rd century BCE, who proved that all the numbers in the form 2^n-1(2^{n-1}) were perfect, where n is some positive integer and that 2^n-1 is a Mersenne prime. A Mersenne prime is a prime number that is one less than a power of 2.

For example, $28 = 2^{3-1}(2^3 - 1) = 4 \times 7$.

We prove this as follows: using our formula for $\sigma(n)$,

[2] Greatest common denominator (gcd)

$\sigma(2^{n-1}(2^n - 1)) = \frac{2^{(n-1)+1} - 1}{1} \times ((2^n - 1) + 1)$

$= (2^n - 1) \times 2^n = 2 \times 2^{n-1}(2^n - 1)$, as $2^n - 1$ is a prime (so $\sigma(2^n - 1) = (2^n - 1) + 1 = 2^n$) and 2^{n-1} and $2^n - 1$ are coprime.
Therefore, all numbers of the form $2^{n-1}(2^n - 1)$ are perfect if $2^n - 1$ is prime.

Proof of the form of all even perfect numbers

Euler, working in the 18th century, proved that all even perfect numbers were of Euclid's form. A proof goes thus:

Let $n = 2^k x$ be an even perfect number, where x is odd and k is a positive integer. 2^k is coprime with x and so $\sigma(n) = \sigma(2^k x) = \sigma(2^k) \times \sigma(x) = 2 \times 2^k x$ given that n is a perfect number. $\sigma(2^k) \times \sigma(x) = (2^{k+1} - 1) \times \sigma(x) = 2^{k+1}x$.

Let $s = \sigma(x) - x$ (this is the sum of the factors of x except x itself, or equivalently the factors smaller than x).

$(2^{k+1} - 1) \times (x + s) = 2^{k+1}x$
$2^{k+1}x - x + s(2k+1 - 1) = 2^{k+1}x$
$x = s(2^{k+1} - 1)$

Therefore, $s \mid x$ and $s < x$ as $2^{k+1} - 1 \geq 2^2 - 1 > 1$.
Therefore, s is a factor of x and being smaller than x, it must have been included in the sum which constitutes s. Therefore, s must be the only factor of x smaller than x, so must be 1 and x must be a prime so that it only has the factors of 1 and itself and is equal to $2^{k+1} - 1$.

Therefore, $n = 2^k(2^{k+1} - 1)$ where $2^{k+1} - 1$ is a prime.

Therefore, all even perfect numbers are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is a prime.

Odd perfect numbers:

The main question we're asking is whether any odd perfect numbers exist. However, it has been shown via computational methods that there are no odd perfect numbers less than 10^{1500}. This does not mean though, that no odd perfect numbers exist. Let me put 10^{1500} into perspective – it's a number so vast that it defies physical reality. If every atom in the observable universe, all 10^{80} of them were each a supercomputer, and they checked 1 trillion numbers each second since the Big Bang (around 10^{18}), they'd have only checked 10^{110} numbers. At that current rate, you'd need 3 x 10^{1400} more years to reach 10^{1500}, a timespan so incomprehensible – just to exhaustively check up to 10^{1500}. That's 10^{1390} times longer than the universe has existed. That's the time by which every star in existence will have died 10^{1300} times over before it's done.

In the 18th century, Leonhard Euler, a Swiss mathematician proved that all odd perfect numbers are in the form:

$$\mathbf{N=p^{4m+1}Q^2}$$

where p is a prime number of the form p=4k+1 and does not divide Q, and m ∈ ℕ, and

Q ≡ 1(mod 2) and is ∈ ℕ.

Euler's form seems arbitrary, until you see how it relates to the algebra of σ(n) Now, let's show why this is the case.

Let n be an odd perfect number. This means that $\sigma(n) = 2n$, as shown earlier. This "2n" must contain exactly only one factor of 2 since n is odd. Now, consider the prime factorisation:

$$n = 3^a 5^b 7^c \dots p_i^{x_{p_i}} \dots$$

where $a, b, c, \dots, x_{pi}, \dots$ are non-negative integers. Since $\sigma(n)$ is multiplicative, it can be expressed as:

$$\sigma(n) = \sigma(3^a)\sigma(5^b)\sigma(7^c) \dots \sigma(p_i^{x_{p_i}})\dots,$$

And each prime power sum is coprime to the others. In order for $\sigma(n)$ to only have only one factor of 2, only one of these sums must be congruent to 2 (mod 4), while all the others need to be odd. Thus, to produce the factor sum ≡ 2(mod 4), we must have one prime ≡ 1(mod 4) to the power of a positive integer ≡ 1(mod 4). The rest of the primes must all be to even powers so that their factor sums are all odd.

Let's consider the special case where $n = q^a R^2$ where q is a prime, $a \in \mathbb{N}$, $q \equiv a \equiv 1$(mod 4) and $R \equiv 1$(mod 2) and ∈ ℕ.

In order to satisfy the two conditions required,

$\sigma(q^a)$ must contribute the single factor of 2: Since q ≡ 1(mod 4), the sum:

$$\sigma(q^a) = 1 + q + q^2 + \dots + q^a$$

consists of an odd number of terms (q+1 terms), which all ≡ 1(mod 4). Summing an odd number of such terms gives a total congruent to 2(mod 4), which means $\sigma(q^a)$ is divisible by 2 but not 4. This precisely holds when a ≡ 1(mod 4).

The other condition is that $\sigma(R^2)$ is odd. Since R^2 contains only even exponents, each factor sum of any square number is an odd sum of an odd number of odd terms. Thus, $\sigma(R^2)$ is odd.

This leads us to the conclusion that all odd perfect numbers have the form $q^a R^2$ where q is a prime, $a \in \mathbb{N}$, $q \equiv a \equiv 1$(mod 4) and $R \equiv 1$(mod 2) and ∈ ℕ. Furthermore, Odd perfect numbers are of the form $q^a R^2$ where q is a prime, $a \in \mathbb{N}$, $q \equiv a \equiv 1(mod\ 4)$ and $R \equiv 1(mod\ 2)$ and ∈ ℕ or in other words a number in the form $4k + 1$ to some power multiplied by the square of an odd number. This is equal to some number of the form $4k + 1$ times another number of the same form. $(4k + 1) \times (4l + 1) = 16kl + 4k + 4l + 1 = 4(4kl + k + l) + 1 \equiv 1(mod\ 4)$. Therefore, all odd perfect numbers are of the form $4k + 1$.

Recently, there are some other conditions that have been proven that all odd perfect numbers must satisfy, such as: All odd perfect numbers must be in the form $n = 1$ mod 12 or $n = 117$ mod 468 or $n = 81$ mod 324, the total number of prime factors of an odd perfect number is at least 101, and the largest prime factor of n is $>10^8$ but less than $(3n)^{\frac{1}{3}}$.

Conclusion:

In conclusion, while the existence of an odd perfect number remains one of the most intriguing concepts in maths, and though we cannot yet prove whether they exist or not, I believe that it is within reasonable assumption that they don't exist. Despite centuries of searching, and the extensive computational firepower being used to try find these numbers, it is reasonable to suggest that there is no odd perfect number at all. In addition, the fact that there are so many constraints on the form of these numbers, means that there may be more that we haven't been able to find yet. If we eventually find that these conditions are mutually incompatible - hence no number could satisfy all of them simultaneously - that would be the final nail in the coffin.

This situation is not dissimilar to Fermat's Last Theorem, which was believed to be true without a formal proof and was only proven successfully after 350 years. Thus, I believe that there are no odd perfect numbers albeit there is no proof yet of their non-existence. As Euler aptly put it, "Whether… there are any odd perfect numbers is a most difficult question", and given how much time and effort has been dedicated to answering it, makes it seem like this problem will remain unsolved for many years to come.

So, while the odd perfect number's existence remains an elusive mystery, it reminds us that even in the world of the beautiful language of mathematics, some questions may be destined to remain unanswered - at least for now, but perchance it is the shadows themselves – not the light – that is what keeps mathematicians chasing.

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