A Brief History of Calculus

By Duy Anh Nguyen

For neither can live while the other survives

Babylonians actively use mathmetics in their society

The Capulets and Montagues. Harry Potter and Voldemort. In seemingly every field, there is a pair of immortal enemies. And, of course, so does in science. Newton and Leibniz—one died in shame while the other lived in fame. *Well*...at least for some period of time. Before their fallout, Isaac Newton and Gottfried Wilhelm Leibniz were close friends and exchanged letters with each other. However, a dispute over the discovery of an important branch of math led to the downfall of one of these great men: Calculus.

History of Calculus

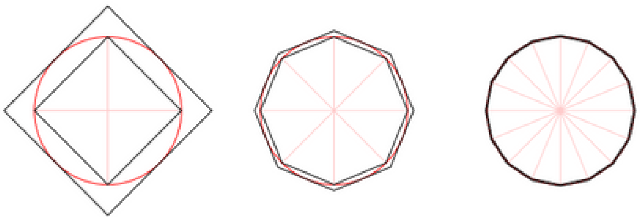
While modern Calculus started with Newton and Leibniz's invention, its origins dated long before that. Countless societies had independently came up with concepts related Calculus. For now, just know that calculus is the study of infinity and the infinitesimals.

<u>Babylon</u>

In 1700 B.C., the Babylonians recognized the existence of a square root of a number. For example, $\sqrt{2}$ is 1.4142135623... While this does not relate to Calculus directly, it signifies how they recognized the concepts of infinity and limits. Later in 300 B.C, believing that the movements of planets, especially Jupiter, were associated with the weather, Babylonians tried to record the instantaneous velocity of the planet as a variable of time. Through that, calculated the total distance travel of the planet—a process similar to integration (which we will soon learn about).

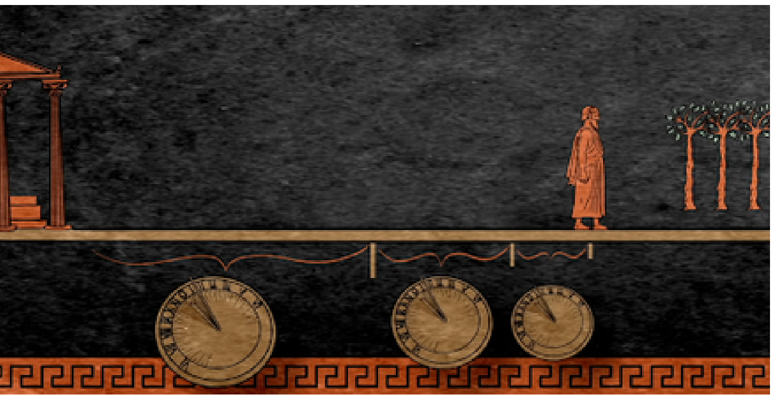
Greece

In Greece, mathematicians and philosophers found the area of a larger shape by cutting it into smaller pieces and finding the area of each individual piece before adding them together. For a circle, the Greeks couldn't, however, use this method. They therefore, used a process called the method of exhaustion. Imagine a square inscribed inside a circle. Then add another angle to it. And then one more. And another one. As the number of angles increases and reaches infinity, the area of the polygon will be approximately the area of the circle. The polygon can now be divided into smaller triangles. Individual areas can then be calculated and added up to calculate the area of a cirlce.



UBC, personal.math.ubc.ca/˜cass/courses/m446-03/exhaustion.pdf. Accessed 31 Jan. 2025.

Additionally, a philosophical statement known as Zeno's paradox during this time also demonstrates the Greek's grasp of early concepts of Calculus. Imagine that you are walking on a mathematical segment from 0 to 1. You walk half of the way to get there. You then walk half of the remaining way. After that, another half of the remaining way. So the process goes like this: (½, ¼,⅛,1/16, ...) The argument is that you will never get to where you want to be, which we know is not true. This forms the argument that the sum of infinite numbers can still add up to a finite number. And actually, ½+¼+⅛+1/16+... does lead up to a finite sum, which is, *voila*, 1.



Source: https://www.youtube.com/watch?v=EfqVnj-sgcc

The series given above is actually called the geometric series, in which for the absolute value of common ratio r < 1, the sum will be finite and equal to $\frac{a}{(1-r)}$, where a is the first term.

India

Astronomer Bhaskara II proved that at the highest point, the instantaneous velocity of the planet is 0. Mathematician Madhava found the infinite series for sine, cosine, and arctan (now more commonly known as the power series). His work, however, went missing despite being commonly referenced by other mathematicians.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\theta^{2k+1},$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\theta^{2k},$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1} \quad \text{where } |x| \le 1.$$

Madhava series (up) and the power series (down)

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{y!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots \\
\sec x &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots \\
\sin^{-1} x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots \\
\cos^{-1} x &= \frac{1}{2}\pi - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 - \frac{5}{112}x^7 - \frac{35}{1152}x^9 + \cdots \\
\tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 +
\end{aligned}$$

Newton and Leibniz

While other mathematicians helped pave the wave, Newton and Leibniz tied all these distinct concepts together. However, they viewed calculus differently. Sir. Isaac Newton saw Calculus as a way to solve physics problems, calling it the method of fluxions. Moving quantities known as fluents changed with respect to time, and the rate of this change was known as fluxions. He relied on geometry to prove his invention. However, Leibniz, through mathematical reasoning, set up the foundation and the notation for the calculus we use today.

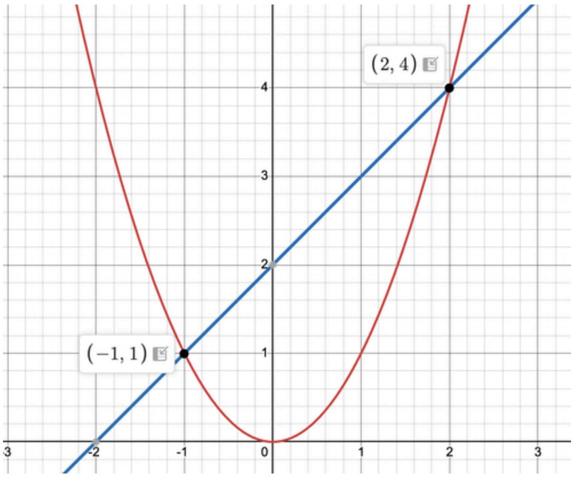
Modern Calculus: Differentiation

Modern calculus concerns itself with two branches: differential and integral calculus. Differential calculus aims to find the instantaneous rate of change of a function. In algebra, we learn of the concept called slope (rate of change) for a linear function, which we can find by the formula:

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

For a linear function, the slope will always be constant. For a nonlinear function, we can not find the rate of change by algebraic means. However, we can find the average rate of change between 2 distant points of a function using the formula of the slope of a secant line. For example, for a function $y = x^2$, you can find the average rate of change between point (-1,1) and point (2,4) as

$$m = \frac{1-4}{-1-2} = 1$$



However, the secant line only shows the average rate of change between two points. What about the average rate of change at a single point? For example, the instantaneous rate of change at x=3 for the graph of y=x^2? Then, mathematicians came up with this formula:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The h in this formula bears a similar function as the $x_1 - x_2$, except in this case, that difference is close to 0 (but not 0). Mathematicians called the instantaneous rate of change **derivative** and the process of finding it **differentiation**. As such, the derivative of $y = x^2$at x=3:

$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \\ &= \lim_{h \to 0} \frac{6h + h^2}{h} \\ &= \lim_{h \to 0} (6 + h) \\ &= 6 \end{aligned}$$

Picture from https://en.wikipedia.org/wiki/Calculus

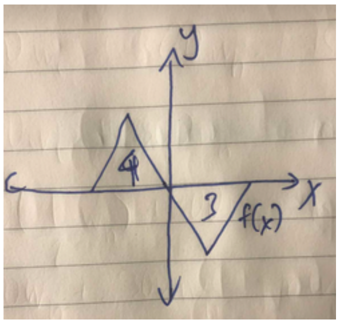
Later, mathematicians realized that there are easier ways to differentiate a function (picture of example below). Here d[a function] /dx is another notation for the derivative. Another common notation is y′ or y prime.

Name	Rule
Power	$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$
Sum	$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
Product	$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
Chain	$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot g'(x)$

Source: https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-2-new/ab-3-5a/a/review-categorizing-functions-for-taking-derivatives

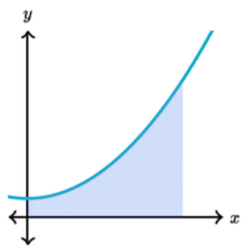
Integral Calculus

Another branch of calculus, integral calculus, concerns itself with finding areas under the curve. It is exactly what its definition suggests. A basic example of finding the integral can be seen here:



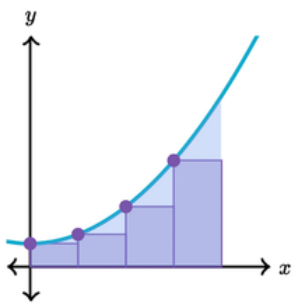
Anything above the x-axis is considered positive, and anything under the x-axis is negative. Thus, the total area under the curve of f(x) in this case is 4-3=1.

The image above is just an easy case of integral calculus. But what about functions like this?

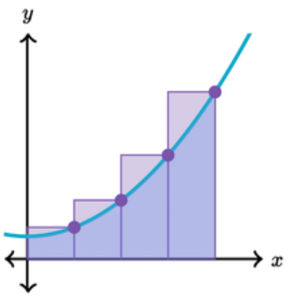


Source: https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-2/a/left-and-right-riemann-sums

There, we have a few choices. A method called Riemann sums suggests that we can divide the curve into multiple rectangles with the top left or top right angles touching the curve. We can find the area of each rectangle (base x height) and then add them up together.

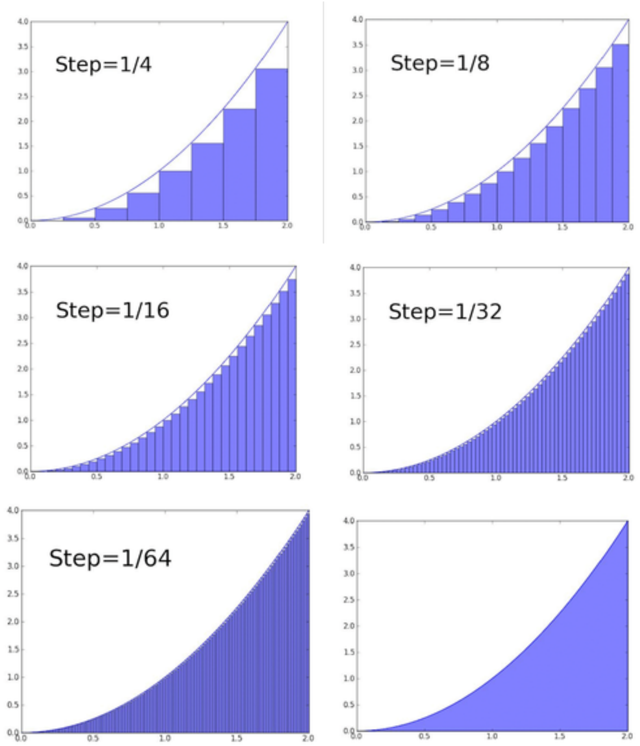


Left Riemann sums



Right Riemann sums

Yet, from the two pictures above, we can see that this method either underestimates or overestimates the area under the curve. But what if we cut the curve into more and more tiny tiny rectangles (hence infinitestimal) so that the rectangles fit more neatly under the curves?



As the number of rectangles increases and approaches infinity, the rectangles better represent the area under the curve.

Mathematicians denote this as $\int_a^b f(x)\,dx.$

f(x) is the height and dx is the base of a infinitesimal triangle. Hence, f(x) dx represents the areas of the rectangle (base x height). The elongated s symbolizes the sum of all areas. a and b represent the lower and upper bound of the x.

Fundamental Theorem of Calculus

Integral and differential calculuses are connected by the fundamental theorem of calculus. This theorem shows that integral is the opposite of derivative, and one can find an original function by integrating its derivative.

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

One of the most famous demonstrations of this relationship is between position, velocity, and acceleration. Velocity is the rate of change of position with respect to time, and acceleration is the rate of change of velocity with respect to time. As such, we can find position by integrating velocity with respect to time, and find velocity by integrating acceleration.

With the fundamental theorem of calculus, we can find the application of calculus in every part of life. In biological science, calculus is used to find the population growth of a species of animals. In finance, calculus is used to calculate the price of zero-coupon bonds. Companies also use calculus in their business. Most notably, Coca-Cola uses calculus to optimize its cans with the least amount of materials!

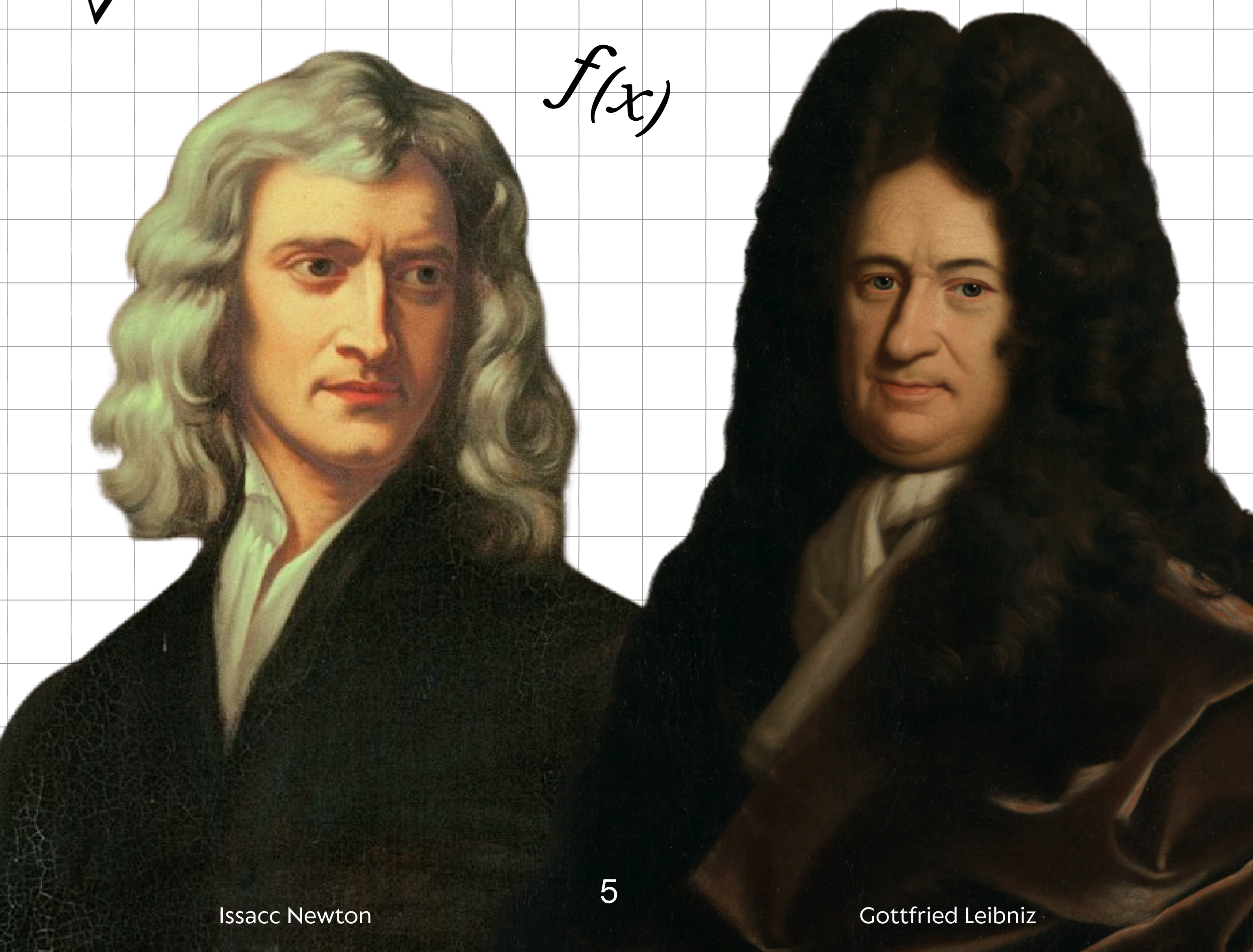
The Newton-Leibniz Controversy

Leibniz first published his findings on differential calculus in 1984, and then integral calculus in 1986. Sir. Isaac Newton, however, did not publish his findings until 1987 with the publication of Philosophiæ Naturalis Principia Mathematica—the book that proved 3 Newton's Laws, gravity, etc. Thus, at first, Leibniz was originally credited as the founder of the field. Angry, Newton decided to taint Leibniz's name, claiming that Leibniz had stolen his idea. He pointed to their letter exchange which did contain his early works on calculus. Since Newton was already known for some of the most important discoveries in science, mathematicians soon took his side. With his influence over Britain's Royal Society (he was also their president), Newton managed to discredit Leibniz, officially known as the father of Calculus. Forever after.

Conclusion

No, just kidding! Of course, Newton was not recognized as the sole father of Calculus. As time reaches infinity, the truth is revealed. Despite the Royal Society's ruling, other countries continued to use Leibniz's method of notation (the notation we use today) due to its efficiency. English mathematicians, with their national pride, refused to do so; however, as English mathematics started to fall behind other countries, they gave up and recognized Leibniz's work in 1820. Later, as historians evaluated the work of both Leibniz and Newton, they recognized that both invented Calculus independently and credited them both as the fathers of Calculus. Finally, the world recognized Leibniz for his work, and Newton for his...ill-mannerism.

As time reaches infinity, the truth is revealed.



Issacc Newton

Gottfried Leibniz