Andy's Problem

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April 6, 2025

One unexpected responsibility you must undertake, being the only "mathy" kid at your high school, is helping your teachers' kids with their math homework—sometimes even three years after you've graduated. It also doesn't help that your literature teacher's 4th grader, Andy, melts your heart every time you see him. This was another one of those evenings when you received yet another call for help. Let's take a look at the image to see what the problem is:

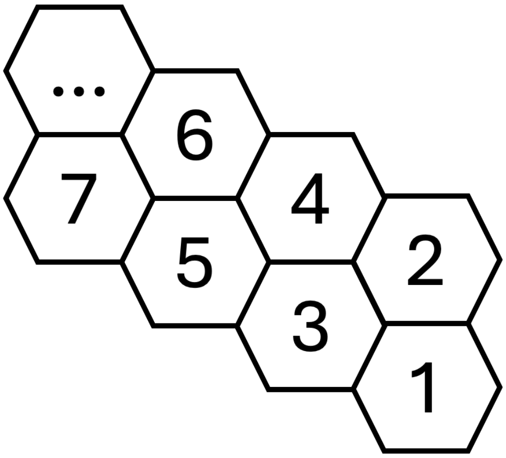


Figure 1: Starting at Hexagon 6, how many ways are there to move to Hexagon 1?

The rules of this problem are as follows: Starting at Hexagon 6, at each step you may move to another hexagon that

*I swear in the name of my father Domingo Montoya, this entire story is true.

1. is directly connected to your current position, **and**
2. has a number smaller than your current position.

For example, from Hexagon 6 you can only move to Hexagon 5 or 4, since you can't move to any Hexagon with a larger number than 6, and Hexagons 3, 2, and 1 are not adjacent to your current position.

The question is: how many ways, following these rules, are there to move from Hexagon 6 to Hexagon 1?

Hmmmmm, you've never seen a question quite like this, but it's a homework problem for a fourth grader, and you're a third-year undergraduate mathematician. Surely this can't be too hard? Still, listing every possible path from 6 to 1 sounds daunting, so let's look at some simpler cases first:

Suppose you start at Hexagon 2. There is only one way to get to Hexagon 1: going directly from Hexagon 2 to 1 in one step, i.e. $2 \to 1$.

If you start at 3, you can go $3 \to 2 \to 1$, or you can go $3 \to 1$ directly, since 3 and 1 are adjacent. That gives us two ways.

What if you start at 4? From 4, the only hexagons you can move to are Hexagon 3 and 2. From Hexagon 3, you have two ways to get to Hexagon 1, and from Hexagon 2, you have one way. Hence, from Hexagon 4 there are $2 + 1 = 3$ ways in total:

$$4 \to 3 \quad \text{(then two ways from } 3 \to 1\text{)}$$

$$4 \to 2 \quad \text{(then one way from } 2 \to 1\text{)}.$$

Now we're getting somewhere. From Hexagon 5, you can only move to Hexagon 4 and Hexagon 3, which have 3 and 2 ways to get to Hexagon 1, respectively. So, from Hexagon 5 there are $3 + 2 = 5$ ways to get to 1.

Doing the same for Hexagon 6, you can move to Hexagon 5 or Hexagon 4, so there are $5 + 3 = 8$ ways to get to Hexagon 1 from Hexagon 6!

Hexagon Position n	F_n i.e. Number of ways to get to Hexagon 1 from Hexagon n
1	1 (i.e. just stay there)
2	1
3	2
4	$2 + 1 = 3$
5	$3 + 2 = 5$
6	$5 + 3 = 8$

Blimey, that's one pattern you didn't expect to see from a primary schooler's homework. For simplicity's sake, let's call F_n as the number of ways to get to Nexagon 1 from Hexagon n. For instance, $F_6 = 8$ because as we discussed above, there are eight different ways to get to Hexagon 1 from Hexagon 6. Let's now analyze our problem with this new notation: at Hexagon n, the only two hexagons you can move to are Hexagon $n-1$ and Hexagon $n-2$, which have F_{n-1} and F_{n-2} ways to get to Hexagon 1, respectively. Therefore, by the same reasoning,

$$F_n = F_{n-1} + F_{n-2}.$$

The next term in this sequence is the sum of the previous two terms: it's the famous Fibonacci sequence (which is why we use F_n)! Named after the Italian mathematician Fibonacci, this sequence starts with $F_1 = 1$ and $F_2 = 1$, then we get $F_3 = F_2 + F_1 = 2$, $F_4 = F_3 + F_2 = 3$, and so on.

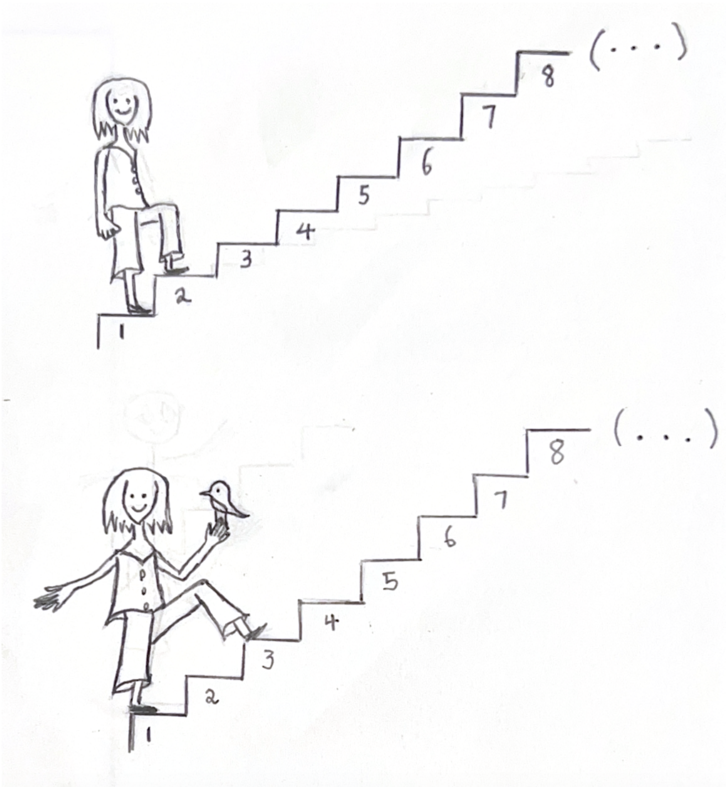
What is great about this way of solving this problem is that it gives us a general solution to this problem. Not only we solve it from Hexagon 6, we can also solve it from, say, Hexagon 10: just find F_{10} by adding your way up.

Before you start contemplating how to teach Andy this famous sequence, you notice something else peculiar:

Starting at Hexagon n, there are really only two possible hexagons you can land on in one step: Hexagon $n-1$ and Hexagon $n-2$ (except at Hexagon 2 and Hexagon 1). So effectively, you're taking the number n and repeatedly subtracting 1 or 2 until you reach 1. Our hexagon question then becomes: how many different ways are there to subtract 1 or 2 from n to end up at 1? Equivalently, starting at 1, how many ways are there to reach n by adding 1 or 2 at each step?

But we already know how many ways there are: F_n ways! We've discovered a new property of the Fibonacci sequence: it's the number of ways to reach n from 1 by adding 1 or 2. To visualize[1] this, imagine a staircase with n stairs, and you are standing at the first stair. Each step, you can either take a normal step climbing one stair, or take a big leap by climbing up two stairs. How many different ways are there for you to reach the top? This question now has a simple solution: there are F_n ways.

[1] credit of picture below: an English student I enlisted to proofread this article.



This is something I've never seen in my (admittedly junior) mathematical career, and it all stems from a homework problem for a 4th grader. If this cool property of the Fibonacci sequence doesn't impress you, at least take away the following:

1. Deep insights often come from simple questions, so never overlook something just because it seems trivial.
2. If a problem is too hard (in mathematics or in life), break it down into simpler parts. If Hexagon 6 seems too daunting, break it down to Hexagon 5 and Hexagon 4. Still too hard? Break it down again.
3. Also, seriously? A question on the Fibonacci sequence (or linear recurrence relations, if you are a maths student) for 4th grade? Cut our kids some slack.