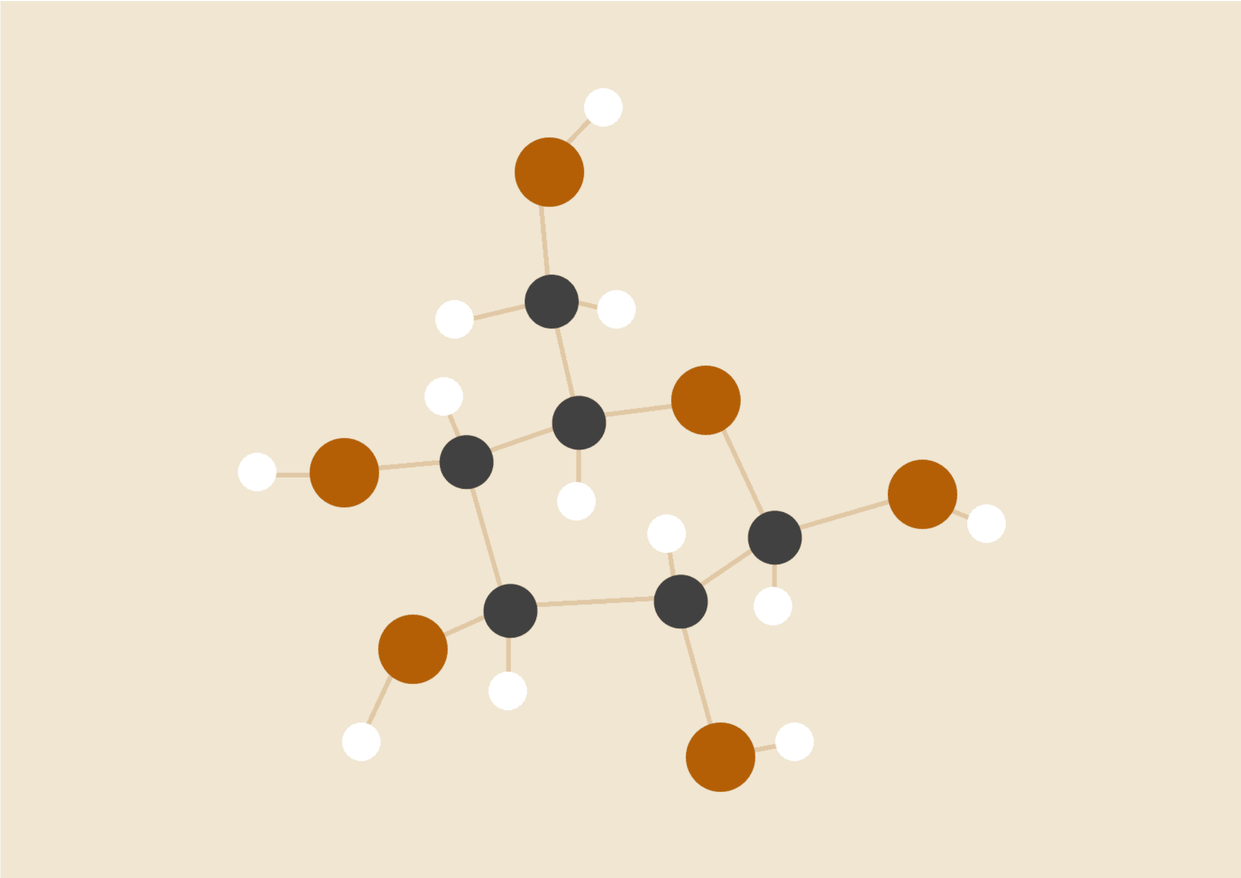
EXPORING INFINITY



By

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Introduction

For this paper, I have decided to discuss and explore some of the exciting and mind-blowing properties of infinity.

I have always found infinity to be an interesting topic. I first encountered infinity at school when I was told there are an infinite number of ways to cut a circle in half and that Pi doesn't repeat and goes on forever. For a young boy, this was both exciting and impossible to wrap my head around! I would like to share my journey of exploring infinity.

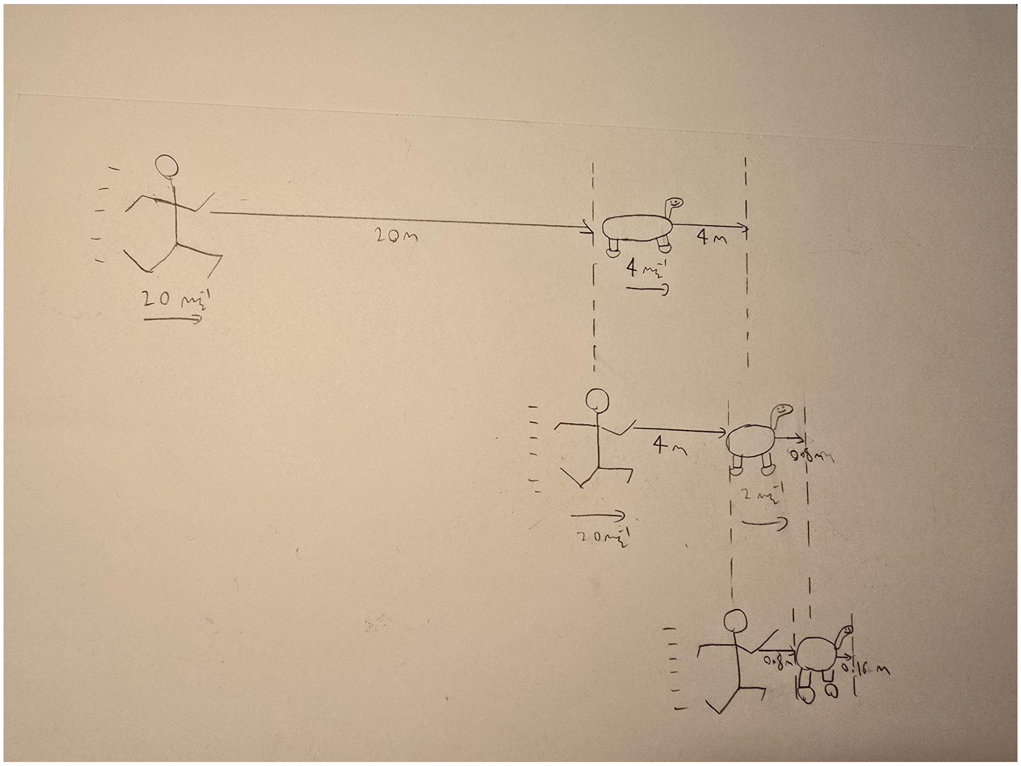
Achilles and the Tortoise

Imagine yourself in ancient Greece around 500 B.C, your friend Pythagoras had just created a theorem to relate the sides of a triangle and everyone is celebrating this great achievement when suddenly a philosopher called Zeno breaks down the door and ruins everyone's mood by proving that motion is impossible. Let's break down what he said.

The philosopher Zeno stated that if the great Hero Achilles were to race a tortoise he wouldn't win if the tortoise was given a head start. This is because every time Achilles caught up with the tortoise, the tortoise would have moved forward in the time that Achillies had run to the point the tortoise had previously been. Zeno said that this would happen again and again forever because there would always be a small distance between Achilles and the tortoise. Therefore, Zeno proposed that Achilles would have had to run an infinite number of smaller and smaller distances and would never reach the tortoise. If taken to the extreme case, this would mean that all movement would be impossible as you would need to take an infinite number of small steps, taking an infinite amount of time to reach anywhere.

However, you think to yourself for a second, "Hold on, this doesn't sound right, I can move". So you set out to disprove this and this is how you do it.

Let's assume that Achilles can run 20ms^-1 and that the tortoise can run at 4ms^-1. I have drawn a diagram to illustrate how this would look and what would happen after Achilles gets to the tortoise's original position.



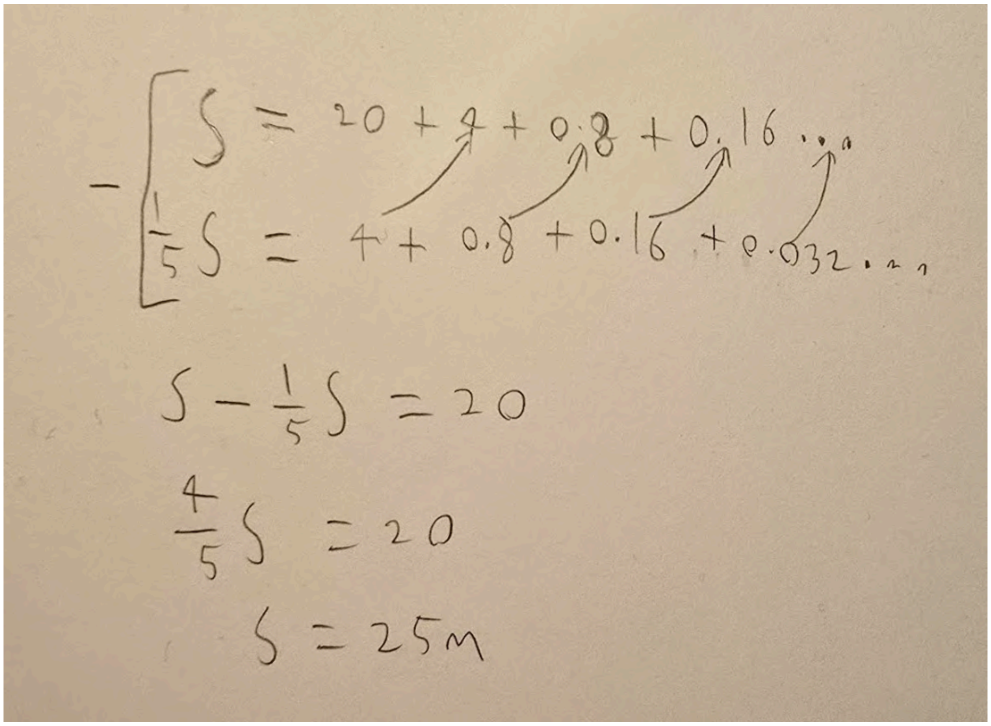
As you can see the distance travelled by Achilles gets smaller and smaller every time so we can write this out as a series labelled S.

$$S = 20 + 4 + 0.8 + 0.16...$$

If you look at each term in the series they seem to have a common ratio of 0.2, as each term in the series is the term before divided by 5. As a smart Greek Philosopher, you take notice of this and divide each term by 5, labelling this ⅕ * S.

$$\frac{1}{5}S = 4 + 0.8 + 0.16 + 0.032...$$

Now you notice a pattern: if you subtract these two series from each other, all terms other than the original 20m cancel out.



This gives you a result that S = 25m and proves Zeno wrong and proves that infinite series can converge. This calls for another celebration.

The Grandi Series

Now let's take a short detour to the early 16th century in Italy, where the priest Guido Grandi was sitting at his desk pondering how to solve a mathematical problem. When Grandi observed the series of numbers 1-1+1-1+1-1… (going on forever), he realised that both 1 and 0 were possible answers to this problem. You may be wondering how that is possible, but let me show you what Grandi did.

First, Grandi saw that you could just group up all (1-1)s and add them together as shown below.

$$(1-1)+(1-1)+(1-1)\ldots$$
$$=0$$

And this result would just be 0

However, if you ordered the numbers in a different way as shown below you would get a result of 1 just by shifting brackets around.

$$1+(-1+1)+(-1+1)+(-1+1)\ldots$$
$$=1$$

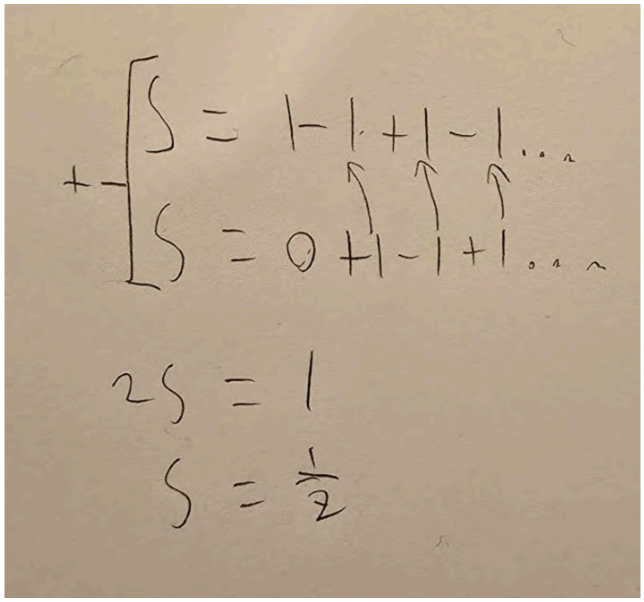
This may already seem interesting, but it is possible to see where these results come from as if you stop anywhere on this series at an even point you would get 0 and at every odd point you would get 1.

How this really gets interesting is when Grandi discovered the third result.
If you recall the binomial expansion of (x+1)^-1 (it is written in the image below)

You can see that the answer looks familiar, and if we take x=1 we can see that the series converges to ½ giving us our third result.

Grandi stated that because the series can give us a result of 0 or ½ (or 1) that God could have created us from nothing.

A simpler way to come to the result of a half is simply by adding a zero to the beginning of the series and we can see in images below how you can add this to the original series and cancel all terms but the first 1.



This leads us to the result that S = ½ and that the series converges at ½

The Sum of all Natural Numbers

Srinivasa Ramanujan was born in India in a town called Erode in 1887. Although he had many health issues throughout his life, his achievements in mathematics were incredible. One of the many things he proved was that the sum of every natural number had a very unusual result. You may think that this is an easy problem and that the answer is just infinite and that the series diverges but Ramanujan would tell you that you are wrong.

We can use the result of the Grandi series we proved earlier to help us with this problem.

First let us think about the series 1-2+3-4+5-6+7… If we use the same trick we used earlier with the Grandi series and add an extra 0 to the beginning of the series, and then add another copy of the original series to the beginning, we get this result.

$$
+\begin{cases} S = 1-2+3-4+5-6\ldots \\ S = 0+1-2+3-4+5-6 \end{cases}
$$

$$
\begin{aligned}
2S &= 1+(-2+1)+(3-2)+(-4+3)+(5-4)\ldots \\
&= 1-1+1-1+1-1+1\ldots
\end{aligned}
$$

Now that looks familiar. It's the Grandi series.

$$
\begin{aligned}
2S &= \frac{1}{2} \\
S &= \frac{1}{4}
\end{aligned}
$$

So we can conclude that this series ends up converging to ¼

Now we can use this to find the sum of all natural numbers. Let’s call the sum of all natural numbers S_1 and let’s call this new series S_2. If we subtract S_2 from S_1 we get.

$$
-\left[\begin{array}{l} S_1 = 1+2+3+4+5+6\ldots \\ S_2 = 1-2+3-4+5-6\ldots \end{array}\right.
$$

$$
\begin{aligned}
S_1 - S_2 &= (1-1) + (2-(-2)) + (3-3) + (4-(-4))\ldots \\
&= 0 + 4 + 0 + 8 + 0 + 12 + 0 + 16\ldots \\
&= 4(1+2+3+4+5+6\ldots) \\
S_1 - S_2 &= 4S_1
\end{aligned}
$$

Which is the same as 4*S_2

So we get the equation:

$$
\begin{aligned}
\frac{-S_2}{3} &= S_1 \\
S_1 &= \frac{-0.25}{3} \\
&= -\frac{1}{12}
\end{aligned}
$$

Which simplifies to S_1 = -1/12, which is just crazy to think about. So the sum of all natural numbers is -1/12!

A Graphical Paradox

Now let's travel to a new place in March of 2025 where a sixth form student is trying to figure out what to write for the last part of his essay. As he was interested in mathematics he had been playing around with trigonometric functions on a graphical calculator called Desmos and came across the function arctan(x)/x. He thought this function looked interesting and similar to the Dirichlet integral (which converges), so he decided to try to integrate it from -infinity to infinity, this is what he did.

First, he decided to convert the function to 2 times the integral of arctan(x)/x from 0 to infinity. We can do this because the function is symmetric and for convenience, we let I = that integral.

$$\int_{-\infty}^{\infty} \frac{\arctan x}{x}\, dx = 2\int_{0}^{\infty} \frac{\arctan x}{x}\, dx$$

$$\text{let } I = \int_{0}^{\infty} \frac{\arctan x}{x}$$

Next we are going to create a function w and let it tend towards infinity. Then we find the limit at 1/w tends towards infinity which would just equal 0 (this is essentially because the bigger the number you put in the denominator, the smaller the number would be). This means we can rewrite the integral as shown below.

$$\lim_{W \to \infty} \left(\frac{1}{W}\right) = 0$$

$$\therefore \quad I = \int_{\frac{1}{W}}^{W} \frac{\arctan x}{x}\, dx$$

Now to evaluate this new integral we can use u-substitution to simplify it as shown below (remember we must change the limits to fit the new integral).

$$\text{let } u = \frac{1}{x} \qquad \frac{dx}{du} = -\frac{1}{u^2}$$

$$x = \frac{1}{u} \qquad dx = -\frac{1}{u^2}\,du$$

$$I = \int_{w}^{\frac{1}{w}} \frac{\arctan\left(\frac{1}{u}\right)}{\frac{1}{u}} \times -\frac{1}{u^2}\,du$$

We can simplify this new integral to just.

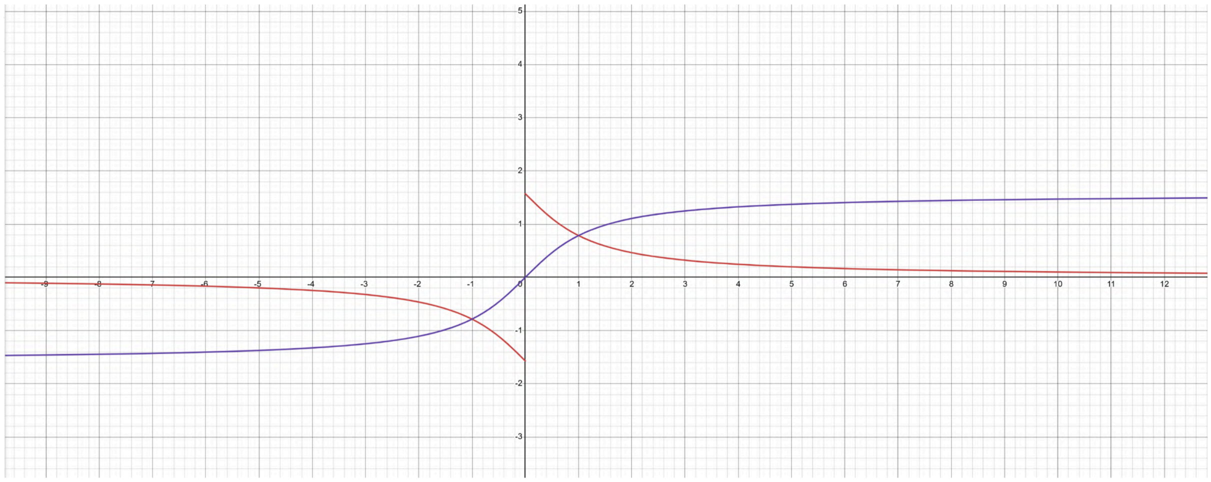
$$\int_{w}^{\frac{1}{w}} -\frac{\arctan\left(\frac{1}{u}\right)}{u}\,du$$

Then what we can do is use the trig identity shown below to rewrite the integral.

$$\left[\arctan\frac{1}{u} = \frac{\pi}{2} - \arctan u \text{ when } u > 0\right]$$

$$\therefore \quad I = \int_{w}^{\frac{1}{w}} -\frac{\pi}{2u} + \frac{\arctan u}{u}\,du$$

We can actually prove this trig identity by looking at the graphs of y=arctan(x) (red) and y=arctan(1/x) (purple).



You can see that if we want to use transformations to make the graphs look the same we have to reflect arctan(x) in the x-axis and then translate it by the vector (0, pi/2) which gives us the identity arctan(1/u) = pi/2 -arctan(u) when u>0.

To carry on with solving the equation we must change the limits by taking out a factor of -1 from the integrand (flipping the limits).

$$I = \int_{\frac{1}{w}}^{w} \frac{\pi}{2u} - \frac{\arctan u}{u} \, du$$

Now we can split this into 2 integrals and solve the equation as we can see the original I again. The integral we are left with is easy to evaluate and simply comes out to pi*ln(w).

$$I = \int_{\frac{1}{w}}^{w} \frac{\pi}{2u}\,du - \int_{\frac{1}{w}}^{w} \frac{\arctan u}{u}\,du$$

$$I = \int_{\frac{1}{w}}^{w} \frac{\pi}{2u}\,du - I$$

$$2I = \int_{\frac{1}{w}}^{w} \frac{\pi}{2u}\,du$$

$$2I = \pi \ln w$$

And with that, all we have to do is let w approach infinity and we get our answer.

$$\int_{-\infty}^{\infty} \frac{\arctan x}{x}\,dx = \pi \ln w$$

$$\pi \lim_{w \to \infty} (\ln w) = \infty$$

$$\therefore \int_{-\infty} \frac{\arctan x}{x}\,dx \text{ diverges}$$

It turned out even though it looked like it converged it didn’t. He decided to plot this function next to another function which he was sure converged, a piecewise function where when x>0 the function would be 1/(2^(x/32)) and when x<0 the function would be 1/(2^-(x/32)).

Let’s try to integrate the piecewise function to see what we get.

The first step is extremely similar to the last integral as we again redefine the integral due to it being symmetrical and we also use u-substitution for this integral. (We also move the constant 32 out of the integral for convenience).

$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{x}{32}}} & x > 0 \\[2ex] \dfrac{1}{2^{-\frac{x}{32}}} & x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 2 \int_0^{\infty} \frac{1}{2^{\frac{x}{32}}}\, dx$$

Let $u = \dfrac{x}{32}$ $\qquad \dfrac{dx}{du} = 32$

$x = 32u \qquad dx = 32\, du$

$$2 \int_0^{\infty} \frac{1}{2^{\frac{x}{32}}}\, dx = 2 \int_0 \frac{32}{2^u}\, du$$

$$= 64 \int_0^{\infty} \left(\frac{1}{2}\right)^u du$$

Now I will use the known results for the integral of a^x to evaluate this integral.

$$= 64 \int_0^\infty \left(\frac{1}{2}\right)^u du$$

$$\left[\int a^x \, dx = \frac{a^x}{\ln a} + c\right]$$

$$64 \int_0^\infty \left(\frac{1}{2}\right)^u du = 64 \left[\frac{\left(\frac{1}{2}\right)^u}{\ln\left(\frac{1}{2}\right)}\right]_0^\infty$$

Now we just need to take the limits as u approaches infinity and u approaches 0 and we are done.

$$\lim_{u \to 0} \left(\frac{\left(\frac{1}{2}\right)^u}{\ln\left(\frac{1}{2}\right)}\right) = \frac{1}{\ln\left(\frac{1}{2}\right)}$$

$$= \frac{1}{\ln 1 - \ln 2}$$

$$= -\frac{1}{\ln 2}$$

$$\lim_{u \to \infty} \left(\frac{\left(\frac{1}{2}\right)^u}{\ln\left(\frac{1}{2}\right)}\right) = 0$$

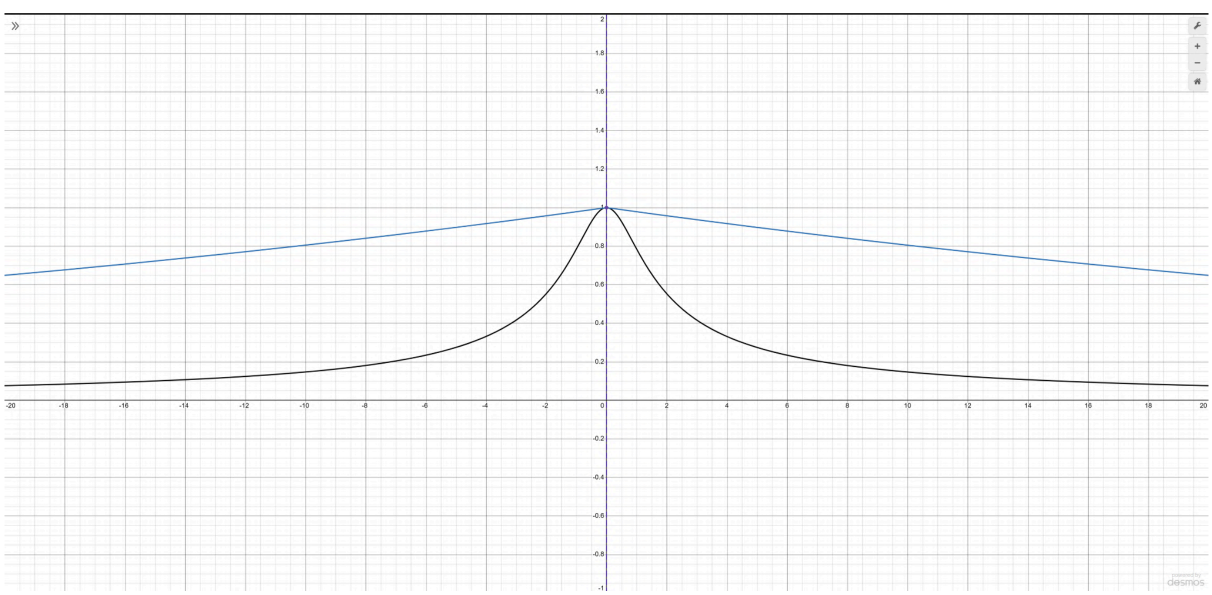
This gives us the result:

$$64 \times \frac{1}{\ln 2} = \frac{64}{\ln 2}$$

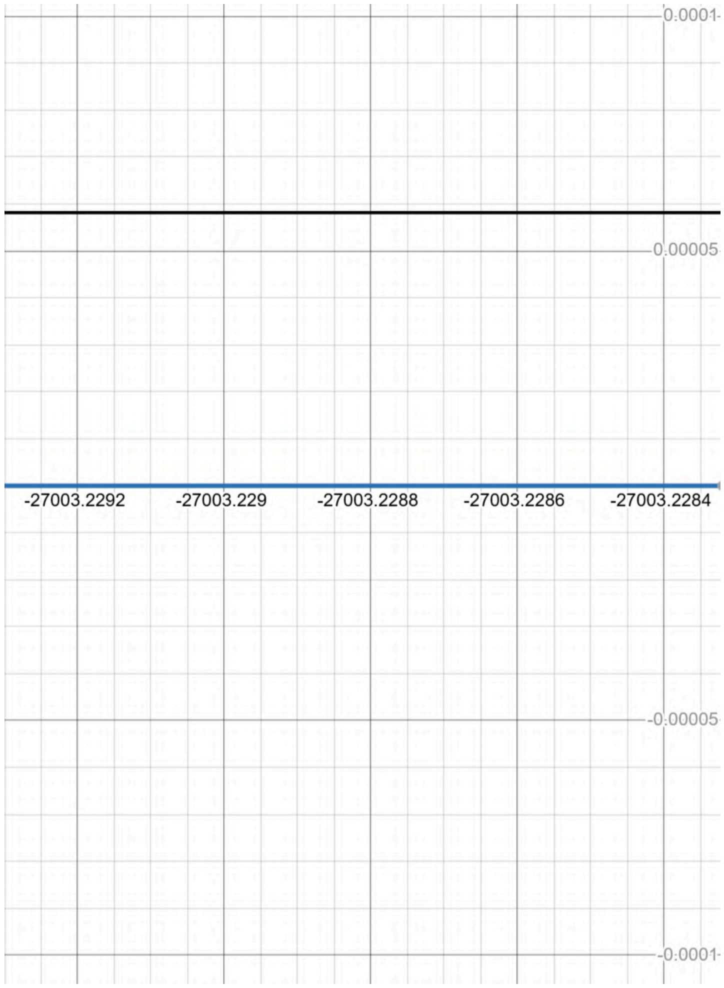
$$\therefore \int_{-\infty}^{\infty} g(x)\,dx = \frac{64}{\ln 2}$$

$$\therefore \int_{-\infty}^{\infty} g(x)\,dx \text{ converges}$$

The reason why the piecewise function converges but the integral of arctan(x)/x between -infinity and infinity diverges is because when you get to extremely high or low values of x the piecewise function decreases much more than the arctan(x)/x function. I just found it so fascinating that sometimes something that looks so obvious at first glance is actually wrong. This is the graph:



When we get to around x=-27000 we start to see that the blue curve (the piecewise function) is lower than the black curve (arctan(x)/x)



Conclusion

Mathematics is a fascinating subject and infinity is a bizarre concept. Our brains can't really comprehend infinity and many interesting thought experiments show how weird it is.

Infinity is used in everything from quantum mechanics to cosmology and is even at the centre of integration and calculating the area under the graph. It's amazing that a concept we struggle to imagine is such a useful tool in the world of mathematics and science.