FIGURE AND GROUND. THE PROHIBITED NUMBERS OF THE COLLATZ CONJECTURE

INDEX

FIGURE AND GROUND

THE PROHIBITED NUMBERS OF THE COLLATZ CONJECTURE

SONATA FOR UNACCOMPANIED TURTLE

The telephone rings and Tortoise picks it up

Tortoise: Hello, this is Tortoise.

Tortoise: Oh, hello, Mr. Achilles, how are you?

Tortoise: A severe headache? Oh, I am sorry to hear it. Do you have any idea what caused it?

Tortoise: Are you working on an unsolved math problem, and you cannot sleep? You have to relax, Mr. Achilles! Life is not just about math! What are you saying?! Do you also have a torticollis? That is incredible, Mr. Achilles! All things happen to you! Do you have any idea what caused it?

Tortoise: Well, no wonder it is stiff, then. All these hours, you are completely crazy! You have lost your mind!

Tortoise: A guitar? What is it doing among all animals? It is the most unexpected object you could find among all these creatures. By chance, Mr. Achilles, do you like music? Do you play any instruments?

Tortoise: All right! All right! Keep calm; I am not getting distracted! There is no need to get upset. Tell me about this dilemma that afflicts you so much!

Tortoise: One of the most intriguing unsolved problems in mathematics…interesting… Can you tell me kindly the name of this dilemma?

Tortoise: What are you saying? It has several names? Well, then tell me at least one…

Tortoise: Ah! I understand! Sorry, you could have told me right away that you were talking about the Collatz Conjecture! Huh… You have found a nice puzzle, Mr. Achilles… "Mathematics may not be ready for such problems", Paul Erdős said. However, you have to apologize to me. At my venerable age, typical of my species, I am losing my memory skills. I beg you to explain the whole riddle again; this time, I will not be distracted.

Tortoise: In a nutshell, if I understood enough, take a natural number. If the number is even, divide it by two. If it is odd, multiply it by three and add 1. And then?

Tortoise: I cannot believe that! If we repeat these steps, all numbers will reach one if the conjecture is true! You have completely surprised me! However, in light of my innate mathematical rigor, I must ask you for valid proof before I can accept its veracity.

Tortoise: Now you are making fun of me! What do you mean it does not exist?!

Tortoise: Huh…now I understand…You have chosen a very complex riddle!

Tortoise: How many attempts have you made? What?! Please explain this to me better, Mr. Achilles.

Tortoise: Yes, maybe with the help of a metaphor, I can understand better…

Tortoise: Another riddle?! Are you obsessed, sir?! A word with the letter 'A', 'D', 'A', 'C', consecutively inside it…What about "abracadabra"?

Tortoise: Oh, you are right. In this case, "ADAC" occurs backward, not forwards. Now you are tiring me out! I will give you a nice puzzle: see if you can find a word that begins and ends with HE.

Tortoise: That is very ingenious, but that is almost cheating. It is certainly not what I meant! I should have specified the exclusion of "degenerate" solutions. Keep trying; I have a brilliant solution in my head! However, I am still having difficulties with your "ADAC".

Tortoise: It is 8:30 PM, Mr. Achilles. Are you feeling tired yet? Do not worry if you have to go to sleep, I hope your torticollis and severe headache will pass soon! See you tomorrow!

SHORTLY LATER, ACHILLES CALLS THE TORTOISE

Tortoise: Congratulations! Now you will be able to sleep! Unfortunately, I am still in the dark about your "ADAC".

Tortoise: All right. This time, I am going to get a little help.

Tortoise: Yeah, of course, I know the Mosaic II. I have all of Escher's works. I have been to every one of his shows, all over the world.

Tortoise: Yeah, I see the black animals.

Tortoise: Of course, also the "negative space" which defines the white animals.

Tortoise: Oh, that is what you mean when you say "figure" and "ground". What does that have to do with the "ADAC" riddle?

Tortoise: Excuse me, Mr. Achilles! All is too complicated for me! I think you are infecting me with your severe headache!

THE NEXT DAY

Tortoise: Hello! Hello! Mr. Achilles! I got it! I solved your problem!

Tortoise: What do you say? Do you have a new idea for studying the Collatz Conjecture? "Figure" and "ground"? Mosaic II? The "prohibited numbers"? I am curious to know!

INTRODUCTION TO THE COLLATZ CONJECTURE

Mr. Tortoise has told us the primary way to look at the Collatz Conjecture by the following algorithm:

- Choose a number from the natural ones
- If the number is even, divide it by 2
- Else multiply by three and add 1
- If the conjecture is true, reiterating the steps will bring you to one; no matter which number you started.

We can formally represent the algorithm by a function that I called the Collatz Function, in symbol C(n):

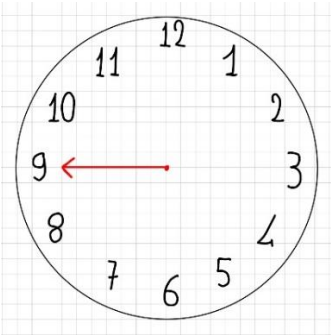
$$C(n) = \begin{cases} \frac{n}{2}, & if\ n \equiv 0\ (mod\ 2), = \ if\ n\ is\ even \\ 3n+1, & if\ n \equiv 1\ (mod\ 2), = if\ n\ is\ odd \end{cases}$$

You are probably thinking about this symbol: (mod 2). What does it mean?

It is called "modulus" and is included in modular arithmetic.

How does modular arithmetic work?

It is also known as "clock arithmetic". A twelve-hour clock is a perfect example of it.



The clock hand is at nine ò clock; if we let 8 hours pass, the clock will reach five ò clock, but 9+8 is not equal to 5. What we are doing with the clock is working on arithmetic modulo 12, which means that after you arrive at 12, you have to start again from 1. In fact, 9+8=17, 17/12=1 remainder 5. That is why nine ò clock + 8 hours equals five ò clock; you must ignore how many times 12 entered in the number and only care about the remainders.

Let's consider another approach to defining the Collatz Conjecture: consider the inverse of the function C(n).

Instead of starting from an arbitrary natural number and ending at 1, as we saw before with the C(n) function, the inverse one will start from 1 and have to reach all the natural numbers if the conjecture is true.

I am going to define the "inverse function" in symbol I(n):

$$I(n) = \begin{cases} \left\{\dfrac{n-1}{3}, 2n\right\}, if\ n \equiv 4 \pmod 6 \\ 2n, \quad if\ n \equiv 0,1,2,3,5 \pmod 6 \end{cases}$$

To validate the Collatz conjecture, we must prove that the inverse function will reach all the natural numbers.
In a nutshell, we have reformulated the problem.

2. SONATA FOR UNACCOMPANIED TORTOISE

The telephone rings and Mr. Tortoise picks it up

Tortoise: Hello, this is Turtle. Mr. Achilles, I am still curious about your ideas on the Collatz conjecture! I have waited for three days!
Tortoise: Mr. Achilles?
Tortoise: Mr. Achilles?
Tortoise: What?! Your best idea turns out to be completely wrong?! Do you have to start all over again?!
Tortoise: Mr. Achilles! You are one of the strongest warriors in history! Don't give up now. Please share your idea with me so we can find a solution.
Tortoise: What mistake have you made?
Tortoise: Ah! Did you define the inverse function incorrectly?

$$I(n) = \begin{cases} 2n, \quad if\ n \equiv 0,2 \pmod 3 \\ \dfrac{n-1}{3}, if\ n \equiv 1 \pmod 3 \end{cases}$$

Tortoise: Why is this definition incorrect?

Tortoise: Ah! You have found numbers making “paths” that cannot be followed backward…and called them “Prohibited numbers”… Can you send me a photo of the diagram you made?

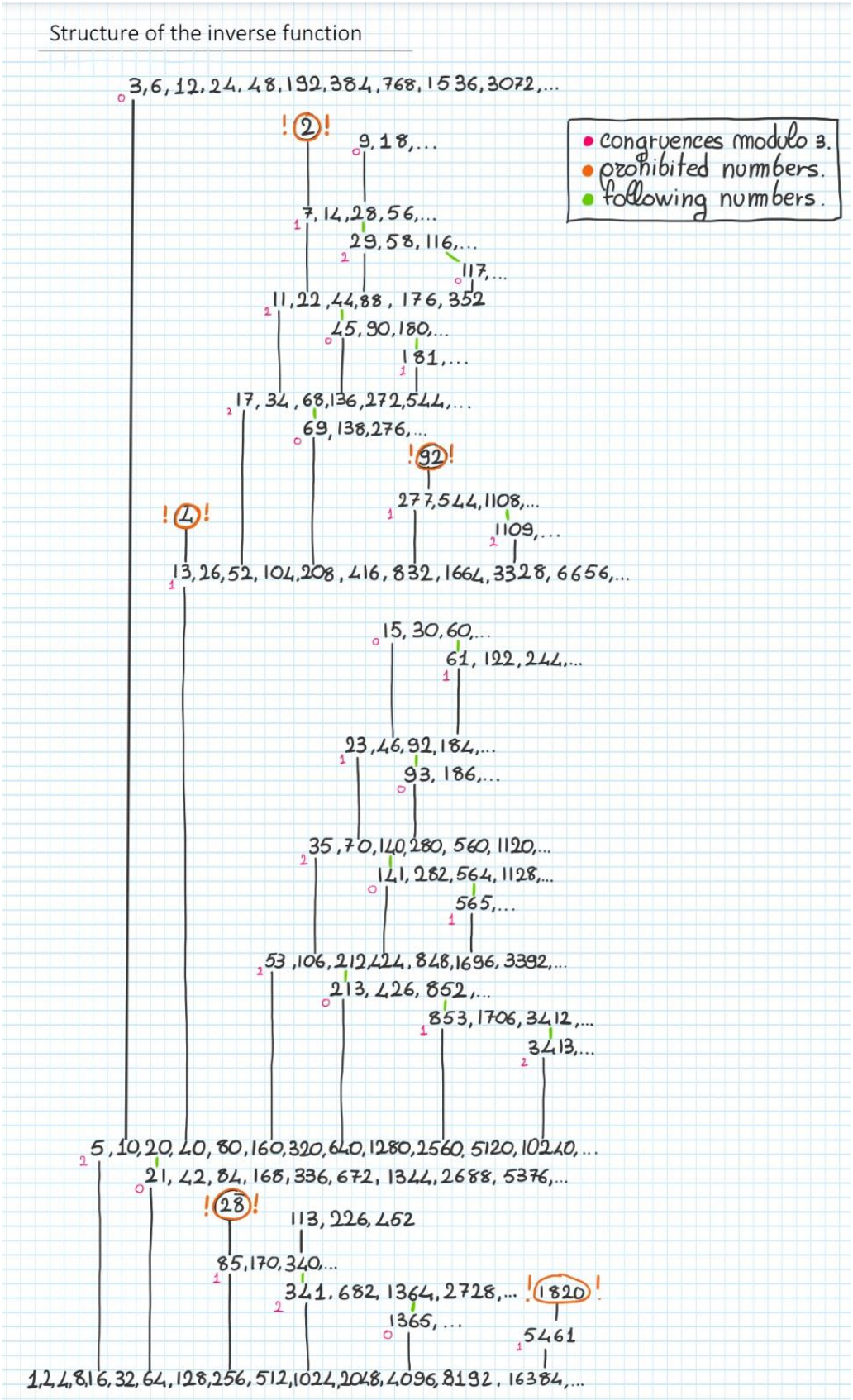
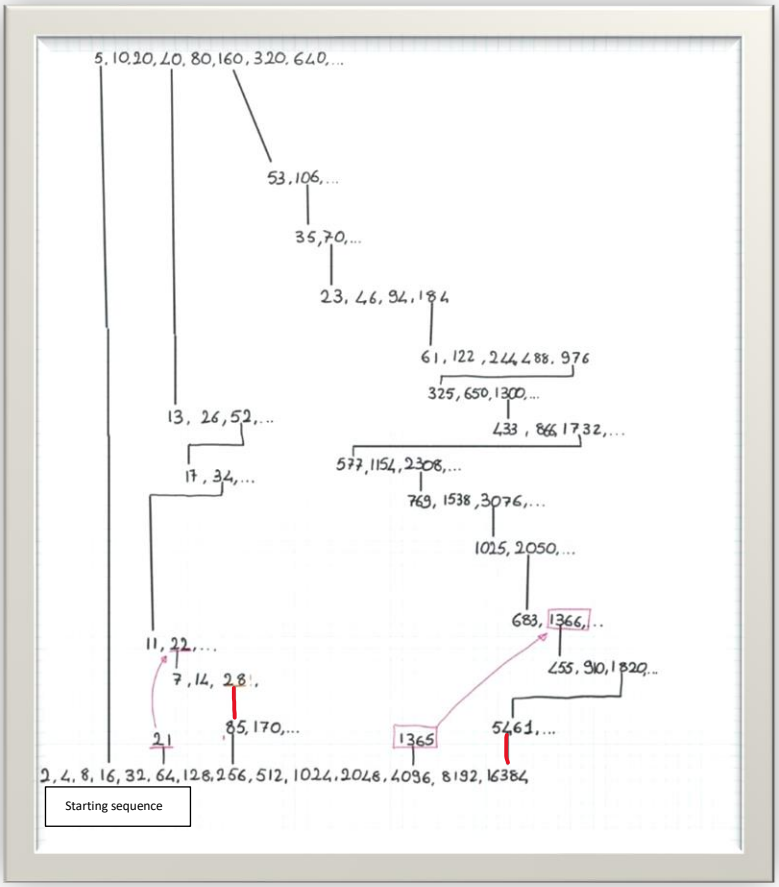


PHOTO 1)

Tortoise: So, the “prohibited numbers” are the ones you have marked with orange?
Tortoise: Yeah, I see… The “prohibited numbers” appear more than once in the function…
Tortoise: In conclusion, you have connected the numbers that appear twice or more, so there will not be repetitions…

PHOTO 2)

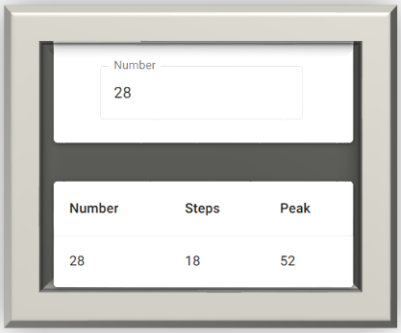


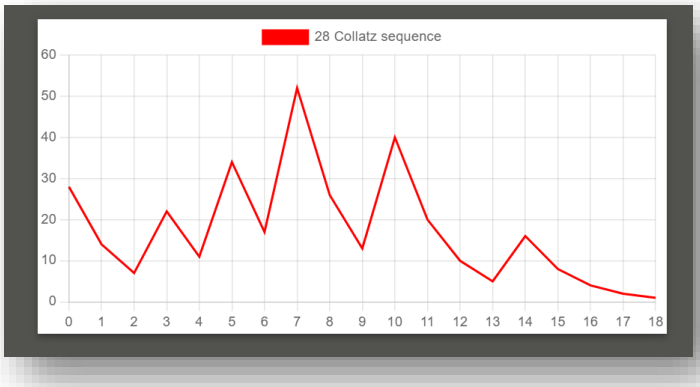
Tortoise: Wow! It is interesting! If we connect the numbers, a regular structure appears! So, all first numbers of the sequence starting from a number divisible by three have their following in the following sequence, and all after the same number of steps!
Tortoise: Oh, you have noticed another thing?
Tortoise: All prohibited numbers generated by the “starting sequence” have their peak after the same number of steps! That’s really strange but at the same time outstanding!

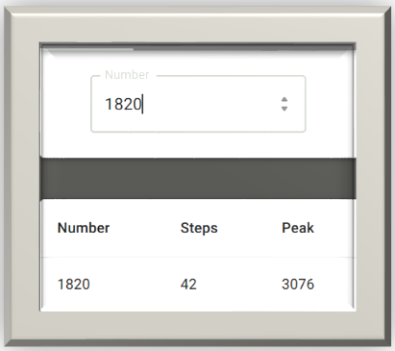
28	1820	116508	7456540	477218588	30541989660	1,95469E+12
14	910	58254	3728270	238609294	15270994830	9,77344E+11
7	455	29127	1864135	119304647	7635497415	4,88672E+11
22	1366	87382	5592406	357913942	22906492246	1,46602E+12
11	683	43691	2796203	178956971	11453246123	7,33008E+11
34	2050	131074	8388610	536870914	34359738370	2,19902E+12
17	1025	65537	4194305	268435457	17179869185	1,09951E+12
52	3076	196612	12582916	805306372	51539607556	3,29853E+12
26	1538	98306	6291458	402653186	25769803778	1,64927E+12
13	769	49153	3145729	201326593	12884901889	8,24634E+11
40	2308	147460	9437188	603979780	38654705668	2,4739E+12
20	1154	73730	4718594	301989890	19327352834	1,23695E+12
10	577	36865	2359297	150994945	9663676417	6,18475E+11
5	1732	110596	7077892	452984836	28991029252	1,85543E+12
16	866	55298	3538946	226492418	14495514626	9,27713E+11
8	433	27649	1769473	113246209	7247757313	4,63856E+11
4	1300	82948	5308420	339738628	21743271940	1,39157E+12
2	650	41474	2654210	169869314	10871635970	6,95785E+11
1	325	20737	1327105	84934657	5435817985	3,47892E+11
	976	62212	3981316	254803972	16307453956	1,04368E+12

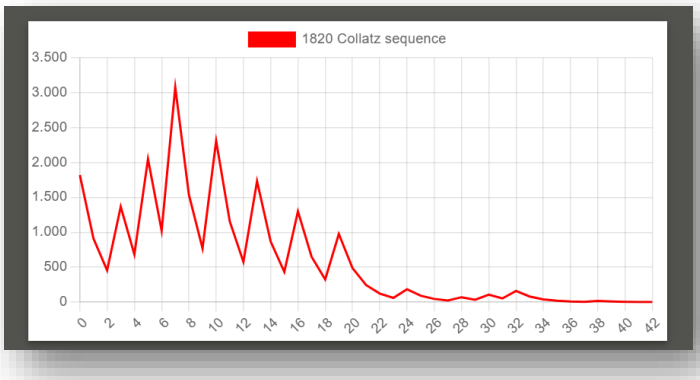
PHOTO 3)

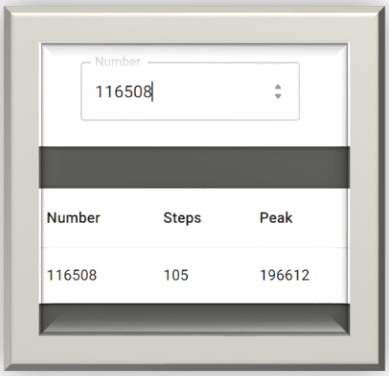
PEAK GRAPHS

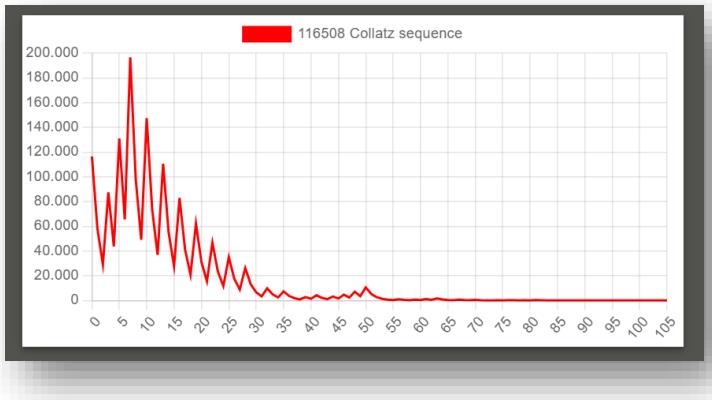


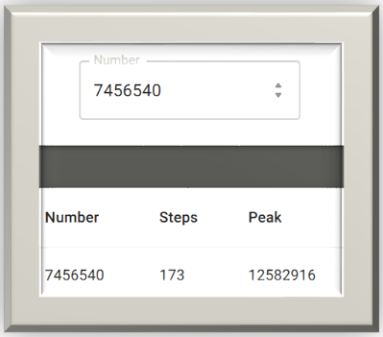


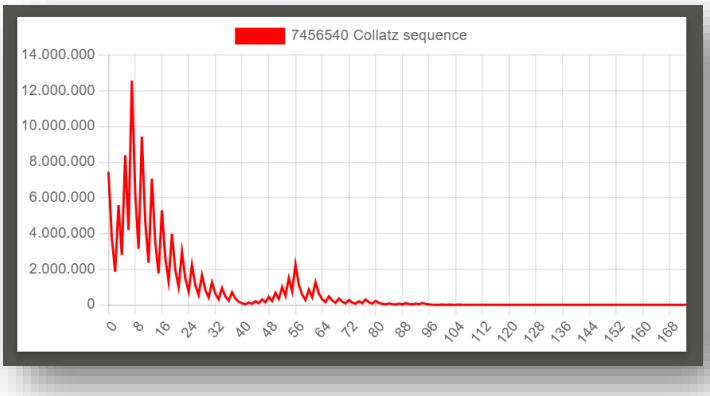


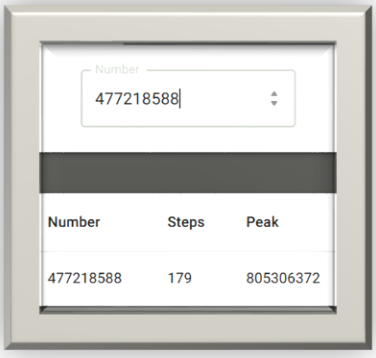


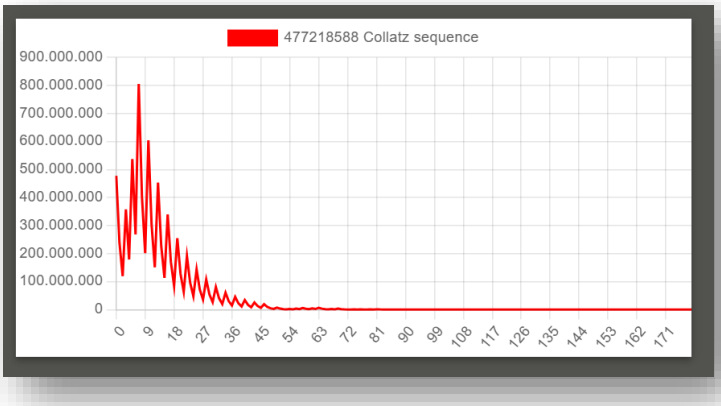

Number
116508
Number Steps Peak
116508 105 196612

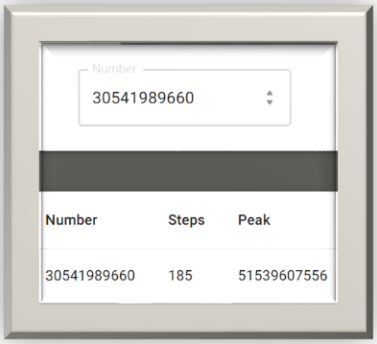

116508 Collatz sequence
200.000
180.000
160.000
140.000
120.000
100.000
80.000
60.000
40.000
20.000
0
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100 105

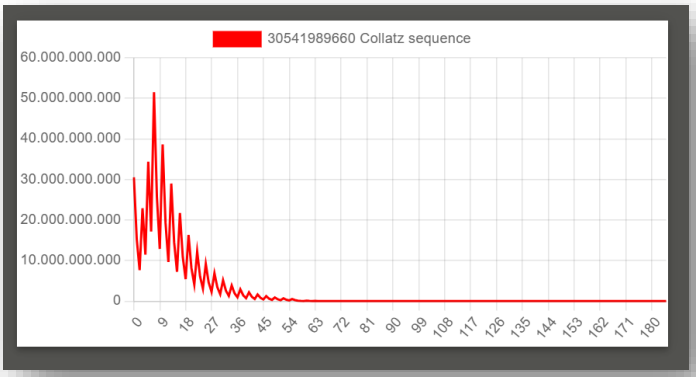

Number
7456540
Number Steps Peak
7456540 173 12582916

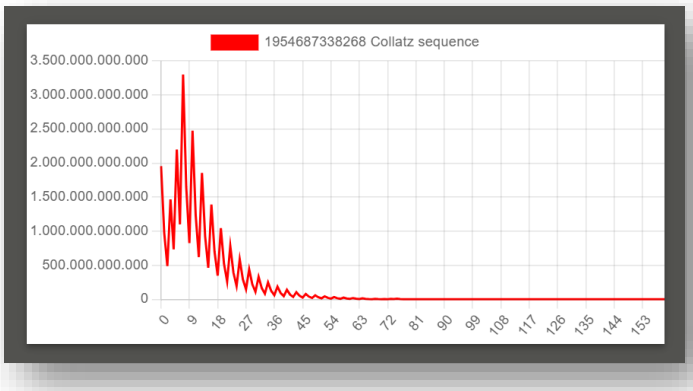

7456540 Collatz sequence
14.000.000
12.000.000
10.000.000
8.000.000
6.000.000
4.000.000
2.000.000
0
0 8 16 24 32 40 48 56 64 72 80 88 96 104 112 120 128 136 144 152 160 168


Number
477218588
Number Steps Peak
477218588 179 805306372


477218588 Collatz sequence
900.000.000
800.000.000
700.000.000
600.000.000
500.000.000
400.000.000
300.000.000
200.000.000
100.000.000
0
0 9 18 27 36 45 54 63 72 81 90 99 108 117 126 135 144 153 162 171







Tortoise: So… you hoped for a proof by induction, right? But you cannot make it because the steps in red, which connect the power of two to the prohibited numbers, are not possible starting from the numbers above the prohibited ones…

Tortoise: Do not get down, Mr. Achilles; they are interesting observations anyway…

ANALYSIS AND OBSERVATIONS

Let us go deeper into the Achilles' mistake. The correct definition of the inverse function is the one in the introduction:

RIGHT: $I(n) = \begin{cases} \left\{\frac{n-1}{3}, 2n\right\}, if\ n \equiv 4 \pmod 6 & 1° case \\ 2n,\ if\ n \equiv 0,1,2,3,5 \pmod 6 & 2° case \end{cases}$

Achilles has defined it as follows:

WRONG: $I(n) = \begin{cases} 2n,\ if\ n \equiv 0,2 \pmod 3 \\ \frac{n-1}{3}, if\ n \equiv 1 \pmod 3 \end{cases}$

The mistake is in using moduli, which significantly impacts the generation of the "prohibited numbers".

Let us take, for instance, 85, which generates the prohibit number 28.

We have to follow the algorithm. The question is: Does 85 in the RIGHT definition belong to the 1° or 2° case?

- If 85 belongs to the first case, then 85≡ $4\ (mod\ 6)$; otherwise, it belongs to the second case.
- 85/6=14 rem 1
- 4/6=0 rem 4
- They have different remainders, then 85 belongs to the second case. In the RIGHT diagram, 85 will not generate 28. In contrast, 85 is allowed to generate 28 in the WRONG definition because it is congruent to 1 (mod 3).

  In conclusion, although Achilles failed to prove the conjecture, he hypothesized that just as in Escher's art, where figure and ground work together making regular patterns, perhaps in the Collatz Conjecture, the truth is not in the numbers we analyze directly but in the hidden structure that governs their behavior.

NOTES BY THE AUTHOR

I structured this article as a chapter of the book "Godel, Escher, Bach" written by Douglas Hofstadter, reinterpreting the dialogue in the chapter three from the viewpoint of Tortoise.
I have applied the relationship between figure and ground to the Collatz conjecture.
Moreover, the dialogue underlines this bond, where hypothesize the sentences that Achilles tells the Tortoise. This is another example of figure and ground: Tortoise's words are the figure and the Achilles'ones are the ground. The same happens in the Collatz conjecture: if we connect the numbers a regular structure appears, and finally, we are able to see patterns only when the figure and ground work together.

REFERENCES

- YouTube, Tom Rocks Math, "How does modular arithmetic work?"
- URL: https://www.youtube.com/watch?v=5OjZWSdxlU0&t=205s
- Wikipedia, "Collatz Conjecture"
- URL: https://en.wikipedia.org/wiki/Collatz_conjecture
- The graphs are screenshots taken by the website:
- Grafico di Collatz
- Mosaic II, Wikioo.org
- URL: https://wikioo.org/it/paintings.php?refarticle=5ZKD4V&titlepainting=Mosaic%20II,%20195