From Abacus to Algorithms: The Mathematical Evolution of Quantitative Finance

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Abstract

This paper provides a comprehensive analysis of the mathematical and computational evolution of quantitative finance, tracing its journey from ancient origins to the AI-driven era. Beginning with early arithmetic and probability theory, we explore pivotal milestones such as the groundbreaking work of Bachelier, Itô, and Black-Scholes-Merton, which formalised stochastic processes and option pricing. The addition of computers revolutionized the field, enabling numerical methods like Monte Carlo simulations, while algorithmic trading introduced new efficiencies and challenges. Today, machine learning and neural networks are transforming risk management, pricing, and portfolio optimization. By examining 5,000 years of history, from 3000 BCE to the 2020's and beyond, this paper highlights the symbiotic evolution of mathematics, technology, and global collaboration in quantitative finance, while also exploring emerging trends such as quantum computing and AI driven finance.

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1 Introduction

I find it quite fascinating to look back at history to observe the then, cutting-edge technologies and innovative ideas brought about by key figures and compare it to what we have today; it brings about a new realm of appreciation for the human race and how we have evolved over time, especially the past century or so. Quantitative Finance is an interdisciplinary field where abstract mathematical theories meets the dynamic realities of financial markets, somewhat enabling us to mitigate risks and take a scientific approach to something so chaotic and random. This little project of mine will also attempt deriving a select few of the slightly complex mathematical equations and attempting to give an intuitive understanding of what these equations and concepts mean as well as their vast applications (even outside of this field of Mathematical Finance). This is going to be a learning experience for both you and I.

2 Ancient Foundations (5000 BCE - 1500 CE)

I think it is interesting to note that the majority of the mathematics taught in high school today are from this era, some even being taught in primary schools; Many foundational concepts such as: Counting, arithmetic, geometry, and even early probability were developed during this time and have been refined over centuries ultimately laying the foundations for probability and by extension mathematical finance.

2.1 Ancient Mesopotamia & Egypt (3000–500 BCE)

This region laid the groundwork for Algebra, Arithmetic and even early computation. They developed a sexagesimal(base-60) system which is still used today in time and angles (360 degrees in a circle and 60 seconds a minute). Babylonian tablets (c. 1800 BCE) show loans with 20% interest which are solved using geometric progression. A tablet from Larsa computes a loan repayment as $1\frac{1}{2} \times 1\frac{1}{2} = 2\frac{1}{4}$ representing compound growth. One of my favourite pieces of Mathematical history from this era is Tablet YBC7289, a Babylonian clay tablet that depicts a sexagesimal approximation for $\sqrt{2}$ and is known as "the greatest known computational accuracy in the ancient world". (Can be see in 'Figure 1')

Figure 1: Tablet YBC7289

2.2 Islamic Golden Age (800–1300 CE)

Al-Khwarizmi often hailed as the "Father of Algebra" laid the groundwork for the algebra we know today. Arguably his most famous and impactful work "Al-Kitab al-Mukhtasar fi Hisab al-Jabr wa'l-Muqabala"(translated as The Compendious Book on Calculation by Completion and Balancing) included systematic methods for solving linear and quadratic equations. An outstanding feat to say the least; his work emphasized practical applications, such as inheritance calculations, land division, and trade, making it highly influential in both the Islamic world (830 CE) and later in Europe through Latin translations. Interestingly, the term 'algebra' was derived from his name 'Al-Jabr' and his name gave rise to the term 'algorithms' signifying his lasting impact in the field of Mathematics and beyond.

During the 9th to 13th centuries, astrolabes played a vital role not only in determining precise times for daily prayers and establishing the direction of Mecca but also in practical applications more relevant to this paper. They were used to track financial transactions, coordinate trade, and assist in agricultural planning. The astrolabe's versatility and ingenuity made it a symbol of scientific and cultural advancement during the Islamic Golden Age. Its evolution laid the groundwork for the development of more sophisticated instruments and methodologies in finance, particularly in the realm of precise timekeeping, a concept that remains fundamental to the application of Mathematical Finance today ('Figure 2' depicts an astrolabe).

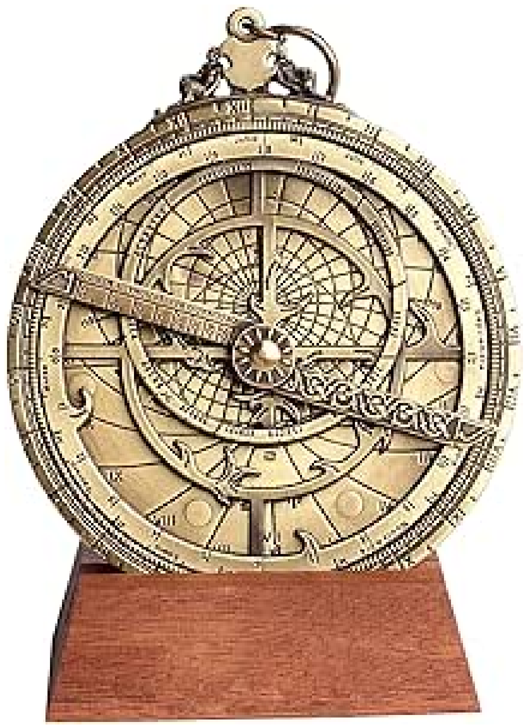

Figure 2: An Astrolabe

2.3 Renaissance Europe (1300–1500 CE)

Luca Pacioli (1494) the "Father of Accounting" significantly enhanced financial record keeping with is book "Summa de Arithmetica, Geometria, Proportioni et Proportionalità", he presented the principles of double-entry book-keeping. This system records both debit and credit entries for every financial transaction, ensuring balance and accuracy. Which you could already guess is invaluable data for the work we do today. Luca Pacioli embedded algebraic logic into book-keeping through the systematic structure of double-entry accounting via the duality concept whereby each transaction has 2 accounts (debit and credit) mirroring the balance of equations as a change in one side must be reflected in the other but inversely. Algebraic Methods can also be seen via the journals which were used to track transactions showing algebraic logic by systematically tracking variables (accounts) and ensuring their relationships remain consistent. Naturally, this system allowed for relatively easy error detection as discrepancies would indicate mistakes; easily comparable to algebraic checks for consistency of equations.

Gerolamo Cardano, a 16th-century polymath (a person whose expertise covers a variety of fields), made significant strides in probability theory while analysing gambling outcomes. In his book "Liber de Ludo Aleae" (The Book on Games of Chance), he formulated some of the field's basic ideas more than a century before the better-known correspondence of Pascal and Fermat. Cardano's insight went far beyond gambling, his book covered important concepts such as expected values, sample spaces and fairness in games; now central to risk modelling and decision making under uncertainty.

Sample Spaces
In probability theory, a sample space simply refers to the set of all possible outcomes
For example:

- Flipping a coin will have the sample space {Head, Tails}
- Rolling a 6 sided die will have sample space {1,2,3,4,5,6}
- Flipping two coins will have sample space {HH, HT, TH, TT}.

Expected Values
The Expected Value refers to the average or the long term value of a random variable determined by its outcomes and their probabilities. It's computed as:

$$E(x) = \sum_{i=1}^{n} P(x_i).x_i$$

Where:

- $E(x)$ is the Expected Value of a random variable X,
- x_i represents the possible outcomes,

- $P(x_i)$ represents the probability of the x_i variable occurring.

I find it interesting that the conventional definition of probability is somewhat auto-logical where the probability of a given event is seen as the expected proportion of said event occurring (from a sample space) given all events are independent, mutually exclusive and **equally likely**. This uses the concept of probability in its definition.

To formalize the idea:

- let Ω be the set of all possible outcomes of a random experiment,
- let N denote the number of items in the sample space
- Then we can define the sample space as: $\Omega = \{w_1, w_2, w_3, ..., w_n\}$
- For any outcome (w_i) in the sample space, assuming equal likelihood, the probability of (w_i) can be expressed as:$P(w_i) = \frac{1}{N}$

UPDATE: This definition of probability is no longer widely accepted. Instead, the axiomatic or frequentist approaches are now commonly employed to address such issues. The frequentist perspective defines probability based on the concept of long-term outcomes, where the relative frequency of an event aligns with its probability distribution.

The axiomatic approach to probability, formalized by Andrey Kolmogorov in 1933, provides a rigorous mathematical foundation. It defines probability as a set function satisfying specific axioms.
It's derivation is as follows:

Setup:

- Let Ω be the set of all possible outcomes
- Let $\mathcal{F}$ be the set of events (set of subsets of Ω)
- $P : \mathcal{F} \rightarrow [0, 1]$
- A is an event in $\mathcal{F}$

This function is denoted as:

$$P : \mathcal{F} \to [0, 1]$$

The function P must satisfy the following axioms:

1. **Non-negativity:** For every event $A \in \mathcal{F}$,

$$P(A) \geq 0$$

2. **Normalization:** The probability of the entire sample space is 1, i.e.,

$$P(\Omega) = 1$$

3. **Countable Additivity:** For any countable sequence of mutually exclusive (disjoint) events $A_1, A_2, A_3, \dots \in \mathcal{F}$, the probability of their union equals the sum of their individual probabilities:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Went on a little bit of a tangent there but back to the main task at hand.

3 Stochastic Revolution (17th–19th Century)

Ladies and Gentlemen, it is finally the moment that we have all been waiting for. The fun stuff has begun :). This is the era where most of the stuff taught in introductory University Modules has originated from and includes the magnificent insight from many outstanding individuals in shaping the field of Quantitative Finance that we know today. In this period, the basics for probability theory will be established and we get to see a glimpse of one of the first Financial Models for stock prices.

3.1 Probability Theory (1600s - 1800s)

In 1654, Blaise Pascal and Pierre de Fermat engaged in a groundbreaking exchange of letters that laid the foundation for modern probability theory. These were sparked by problems related to gambling, particularly the "problem of points", which involved determining how to fairly divide stakes in an interrupted game of chance. Pascal and Fermat made the use of combinatorics to calculate probabilities, making use of the idea of counting all possible outcomes to assess likelihoods.

The problem of points:

The problem of points explores how to fairly divide the stakes when a game of chance is interrupted and cannot be concluded. An example case:

Imagine 2 players are participating in a game to win an amount of money contributed by both players whereby they take turns and earn points every round and the winner takes all. In order to win the prize money, they need to reach a predetermined number of points. However, The game has been interrupted before either player gets the predetermined number of points. The question is "How should the prize money be distributed?" Example Scenario:

- Player A needs 3 points to win
- Player B needs 2 points to win
- The game has been Interrupted

Pascal and Fermat's Solution:

They considered all possible future outcomes and distributed the pot.

- First calculate the probability of each player winning should the game continue.
- Then divide the prize money according to that probability

Significance:
This problem not only built on key concepts like expected value and combinatorial probability, but it also demonstrated how mathematics could provide rational solutions to real-world fairness problems. It was foundational in the development of modern probability theory.

Jacob Bernoulli's work in 1713 introduced the Law of Large Numbers (LLN) in his book, "Ars Conjectandi" published after his death. This groundbreaking principle quantifies how outcomes observed over many trials converge towards their true probability (This is the frequentist approach to probability), offering a foundation for statistical reasoning.

The Law of Large Numbers states that as the number of independent trials of a random experiment increases, the average result of those trials approaches the expected value. In practical terms, this essentially means that uncertainty/randomness reduces the more we observe it thus leading to greater predictability in the long run.

Applications in Finance:
The LLN is used for risk diversification.

- By spreading investment across many assets, the idiosyncratic risks (risks that are specific to individual assets) tend to cancel out in the long run.
- Over time, the portfolio's performance becomes more stable, aligning closely with its expected return. Mirroring how the occurrence of a result in the long run tend towards it's Expected Value.

  $$\lim_{x \to \infty} \frac{N(A)}{x} = P(A)$$

  Where:

  - x is the number of trials,
  - $N(A)$ is the Number of times event A has occurred,
  - $P(A)$ is the probability of A occurring,

- $E(A)$ is the Expected Value of A.

$P(A)$ is the probability of event A in a single trial. Over x trials, the proportion of occurrences of A, $N(A)/x$, approaches $P(A)$ as x grows large. This relationship connects the number of occurrences (N(A)) to the expected value $E(A)$ in the context of long-term averages.

For example, in a well-diversified portfolio, the random fluctuations of individual assets average out, reducing the overall uncertainty. Bernoulli's LLN thus provides a mathematical framework to explain why diversification is an essential strategy in risk management. His work remains a cornerstone of probability theory, statistics, and financial modelling.

3.2 Early Financial Models(1900)

Louis Bachelier's 1900 thesis, "Théorie de la Spéculation", marked a monumental leap in financial mathematics by introducing the concept of modelling stock prices using Brownian motion. Louis introduced the idea that stock prices exhibit random movement, a radical departure from deterministic models which were the consensus for models at the time. He proposed that the fluctuations in stock prices could be modelled by what we now recognize as a stochastic process; a random process that evolves over time. This groundbreaking work predated Albert Einstein's independent discovery of Brownian motion in 1905, linking stochastic processes to finance for the first time.
Bachelier modelled stock price S_t at time t as:

$$S_t = S_0 + \sigma W_t$$

Where:

- S_0 is the initial stock price,
- σ is a constant representing volatility, measuring the degree of randomness or fluctuations in stock price,
- W_t is a Wiener process which is a mathematical representation of Brownian Motion

In the context if Bachelier's model, these properties mean that stock price changes are random, normally distributed (over small time intervals) and accumulate continuously over time.

What exactly is Brownian Motion?:

Brownian Motion or a Wiener process is a type if continuos-time ransom process. It was initially observed in unpredictable movement of pollen particles in water by botanist Robert Brown, and later Einstein, Robert Wiener and others provided a mathematical theory for this phenomenon. In finance, Wiener processes serve as a fundamental building block for modelling random behaviour over time.
Mathematical Definition of Wiener processes:

1. Initial Value: At time $t = 0$, the motion starts at zero:

$$W_0 = 0$$

2. Independent Increments For any $0 \leq t_1 < t_2...t_n$, the increments $W_{t+s} - W_t$ are independent of each other.

3. Stationary Increments: The distribution of $W_{t+s} - W_t$ depends only on the time of the interval s, and not t (the total time).

4. Normal Distribution of Increments: For any $t > 0$, the increment $W_{t+s} - W_t$ is normally distributed with mean 0 and variance s:

$$W_{t+s} - W_t \sim N(0, s)$$

5. Continuos Paths: The sample paths are continuos functions of t meaning that no jumps or discontinuities occur.

Derivation of Brownian Motion:

To derive Brownian Motion Mathematically, we can construct it step-by-step as the limit if a discrete-time random walk.

1. **Random walk:** Consider a particle that moves in discrete steps, either up of down by a small amount Δx, at equal time intervals Δt. The particle's position after n steps is:

$$X_n = \sum_{i=1}^{n} \Delta x_i$$

where Δx_i are independent, identically distributed random variables (e.g. $\Delta x_i = \pm 1$),

2. **Scaling to Continuous Time:** To approach a continuos motion, we rescale the time interval Δt and step size Δx. Let:

$$\Delta x = \sqrt{\Delta t}$$

   This ensures that the variance grows linearly with time, which matches physical observations of Brownian Motion.

Properties and Applications of Brownian Motion:
Brownian Motion has the properties which were derived above such as its variance and distribution, where for the Wiener process W_t, we have:

$$W_t \sim N(0, t)$$

This means 2 key things:

1. Normal Distribution of Increments: Each increment of the process is normally distributed with mean 0. This gives us a complete description of the likelihood of different outcomes. In other words, we know exactly how the random "jumps" behave statistically (over time), making it possible to compute probabilities of the process taking on particular values (This is fundamentally linked with the law of large numbers or Central Limit Theorem).

2. Variance Growing Linearly with Time: The variance is equal to t, which implies that the spread (or uncertainty) of the process increases as time increases. This linear relationship is crucial because it tells us that over longer periods, the range of possible outcomes widens predictably.

The Variance and distribution properties of Brownian motion are essential because they provide a complete and predictable description of how randomness accumulates over time and has set the foundations for more accurate models for stock prices (some of which will be explored in this paper). They allow us to calculate probabilities, mathematically trace the process, manage risk and even in the development of models used to describe other Markov processes (A process that in only dependant on its current value not its past values) in other fields such as physics and biology.

Limitations of Brownian Motion:

1. One of the more obvious limitations of Brownian Motion is that it assumes that increments are normally distributed with mean 0 and variance proportional to time. This implies that asset prices have symmetric returns with no extreme movements (equal probability is positive and negative changes and is bounded by the tails in the Normal distribution). This is clearly an issue as financial markets usually exhibit "fat tails" (extreme unpredictable events) and skewness where returns are not symmetrical. Market crashes are more common than what the Normal Distribution Predicts.

2. The assumption of path continuity also provides a problem in real life application. This does not account for events like market shocks or major announcement that cause abrupt price changes which are quite common in the real world.

3. The assumption that volatility (σ), is constant over time and in the real world, it is simply not the case. Volatility is dynamic and fluctuated based on market conditions (The volatility of a Market is volatile).

4. The lack of Mean reversion is another limitation of this model as it simply doesn't account for the fact that prices or rates (such as interest rates and commodity prices) tend to revert to a long term average.

5. The possibility of negative values is also a problem. In Bachelier's model ($S_t = S_0 + \sigma W_t$), Brownian motion can produce negative numbers as it doesn't restrict the range of values. In reality it is very unrealistic that asset/stock prices can become negative.

6. Brownian Motion assumes the independence of increments making it a Markov process. This essentially means that the past movements do not influence the future movements which is not the case in real financial markets whereby the past volatility patterns influence the future movements (This concept is known as Autocorrelation).

7. Over longer time horizons, the Brownian motion model leads to increasingly wide ranges of outcomes due to the unbounded growth of variance. While in reality, there are some constraints to how small or large an asset's price can grow.

We return after yet another tangent.

4 Mathematical Rigour Meets Markets (20th Century)

The 20th century represents a transformative era for Mathematical Finance, where rigorous theoretical frameworks and innovative tools emerged to revolutionize financial analysis. Building upon the foundational concepts introduced by Louis Bachelier and Jacob Bernoulli, this period witnessed groundbreaking advancements in portfolio theory, option pricing, and stochastic calculus. Many of the methods and models developed during this era remain fundamental to modern quantitative finance and are continually refined to address the emerging challenges and complexities of financial markets.

4.1 Modern Portfolio Theory (MPT)

Harry Markowitz revolutionized investment theory by introducing mean-variance optimization, a mathematical framework to balance risk and return in portfolio selection. This foundational work laid the groundwork for Modern Portfolio Theory (MPT) which remains a cornerstone in modern finance.

Key Concepts:

1. Risk and Return:
    - The Expected Return (μ_p) is the weighted sum of individual asset returns.
    - Portfolio Risk (variance of the returns) accounts for both the variability of individual assets and the covariance between assets.

2. Optimization Goal:

    - Minimize portfolio variance for a given expected return. In other words, maximize your expected return for a given level of risk.

The Mathematical formulation for portfolio optimization:
We assume a portfolio of n assets with the following:

- w: An n x 1 vector of asset weights
- μ: An n x 1 vector of expected return for each asset
- Σ: The n x n covariance matrix of asset returns
- μ_p: The target expected return of the portfolio
- Constraints: The sum of the weights must equal 1 ($w^T 1 = 1$) where 1 is a vector of ones.
- R_p denotes the portfolio returns
- R is a vector of random returns for each asset

Formulating the problem:
We wish to optimize a portfolio's returns R_p

$$R_p = w^T R$$

This linear combination represents the overall returns for the portfolio
We also wish to minimize portfolio variance

$$Var(R_p) = w^T \Sigma w$$

The term w^T produces a row vector that presents the weighted covariances and by multiplying by w on the right, we can sum these weighted covariances thus resulting in the overall variance. This form captures both the variance of individual assets and covariances between all pairs of assets as required.
Now let us set up an optimization problem:

We want an optimal portfolio that minimizes risks whilst simultaneously achieving the target expected return (μ_p). Hence, we formulate these constraints:

1. The expected return Constraint: Where the expected return for the portfolio is:

$$w^T \mu,$$

   Where μ is the individual asset's expected returns, We require:

$$w^T = \mu_p,$$

   where μ is the target portfolio return.

2. Full Investment Constraint: We assume that all available funds are invested, meaning that the sum of weights equals 1:

$$w^T 1 = 1,$$

   Where 1 is a vector of ones

Overall formulation: Minimize

$$w^T \Sigma w,$$

subject to

$$\begin{cases} w^T = \mu_p, \\ w^T 1 = 1. \end{cases}$$

Introducing Lagrange Multipliers:
In order to solve a constrained optimization problem, we employ the method of Lagrange multipliers, Introducing Multipliers:

- λ for the expected return constraints
- γ for the full investment constraints

We then set up the Lagrangian as:

$$L(w, \lambda, \gamma) = \frac{1}{2} w^T \Sigma w - \lambda(w^T \mu - \mu_p) - \gamma(w^T 1 - 1).$$

Breaking it down:

1. Objective term: $\frac{1}{2} w^T \Sigma w$ represents the portfolio variance that we aim to minimize and the $\frac{1}{2}$ is included to simplify the derivative later on

2. Penalty for Return Deviation: $-\lambda(w^T\mu - \mu_p)$ penalizes deviations from the expected constraint $w^T\mu = \mu_p$.

   - when $w^T\mu > \mu_p$, this term decreases the objective value
   - And similarly, when $w^T\mu < \mu_p$,, this term increases its objective value.

   Penalty for Full Investment Constraint: $-\gamma(w^T 1 - 1)$ penalizes deviations from the full investment constraint $w^T 1 = 1$.

   - When $w^T 1 > 1$, this term decreases the objective value
   - When $w^T < 1$, This term increases its objective value

Now in order to find the optimum, we must derive the first order conditions by taking the derivative with respect to each value and set them to 0:

- Derivative with respect to w:

$$\Delta_w L = \Sigma w - \lambda\mu - \gamma 1 = 0$$

  This equation defines the balance between the risk and adjustments for return and investment constraints. This equation tells us that at the optimal point, the marginal increase in portfolio risk (given by Σw) is exactly balanced by the adjustments needed to satisfy the constraints. We use the gradient notation because we are working with vectors and matrices and we don't need to write the derivative with respect to all values in the vector or matrix (e.g. rather than $\frac{\partial L}{\partial w_1}, \frac{\partial L}{\partial w_2}, ...$, we write $(\Delta_w L)$

- Derivative with respect to λ:

$$\frac{\partial L}{\partial \lambda} = -(w^T\mu - \mu_P) = 0 \quad , w^T\mu = \mu_P.$$

  This directly recovers the target expected condition.

- Derivative with respect to γ:

$$\frac{\partial L}{\partial \gamma} = -(w^T 1 - 1) = 0 \implies , w^T 1 = 1.$$

  Ensuring that the total investment is fully allocated.

We can now solve these systems of equations to get values for λ and γ: First, from the derivative with respect to w:

$$\Sigma w = \lambda\mu + \gamma 1.$$

Assuming Σ is invertible, solve for w as:

$$w = \Sigma^{-1}(\lambda\mu + \gamma 1).$$

Now, we need to find the multipliers λ and γ using our constraints. To simplify the equations, define three scalars that capture the relationships between μ, 1, and Σ:

$$A = \mathbf{1}^T\Sigma^{-1}1,$$
$$B = \mathbf{1}^T\Sigma^{-1}\mu,$$
$$C = \mu^T\Sigma^{-1}\mu.$$

Applying the Investment Constraint: Plug w into

$$w^T\mathbf{1} = 1:$$

$$1^T w = 1^T\Sigma^{-1}(\lambda\mu + \gamma 1) = \lambda(1^T\Sigma^{-1}\mu) + \gamma(1^T\Sigma^{-1}1) = \lambda B + \gamma A = 1.$$

Applying the Target Return Constraint: Plug w into

$$w^T\mu = \mu_p:$$

$$\mu^T w = \mu^T\Sigma^{-1}(\lambda\mu + \gamma 1) = \lambda(\mu^T\Sigma^{-1}\mu) + \gamma(\mu^T\Sigma^{-1}1) = \lambda C + \gamma B = \mu_p.$$

This gives us a system of two equations:

$$\begin{cases} \lambda B + \gamma A = 1, \\ \lambda C + \gamma B = \mu_p. \end{cases}$$

Solve for λ and γ: Find λ: Multiply the first equation by B to get:

$$\lambda B^2 + \gamma AB = B.$$

Multiply the second equation by A to get:

$$\lambda AC + \gamma AB = A\mu_p.$$

Subtract the first resulting equation from the second:

$$\lambda(AC - B^2) = A\mu_p - B.$$

Therefore:

$$\lambda = \frac{A\mu_p - B}{AC - B^2}.$$

Find γ: Substitute λ back into one of the original constraints, for example, the first:

$$\lambda B + \gamma A = 1 \quad \Longrightarrow \quad \gamma \frac{1 - \lambda B}{A}.$$

Replacing λ gives:

$$\gamma = \frac{1 - \frac{B(A\mu_p - B)}{AC - B^2}}{A}.$$

With a little bit of algebra, this simplifies to:

$$\gamma = \frac{C - B\mu_p}{AC - B^2}.$$

Finally, substitute λ and γ back into:

$$w^* = \Sigma^{-1}(\lambda\mu + \gamma\mathbf{1}),$$

to obtain the optimal portfolio weights in closed form.
Giving us:

$$w^* = \Sigma^{-1}(\lambda\mu + \gamma\mathbf{1}) = \Sigma^{-1}(\frac{A\mu_p - B}{AC - B^2}\mu + \frac{C - B\mu_p}{AC - B^2}1)$$

That took Waay too long...

4.2 The Black-Scholes-Merton Model

The Black Scholes Merton Model is one of the most popular options pricing model used today. Derived by Fischer Black, Myron Scholes, and Robert Merton (1973), it is a partial differential equation (PDE) for pricing options. It makes use of the assumptions of geometric Brownian motion and I will attempt to derive this, first covering the prerequisites such as Geometric Brownian motion and Ito's lemma. Whilst attempting to make each step's relevance in Financial Modelling clear. Wish me luck...

1. **Geometric Brownian Motion (GBM)**
   We begin by modelling a stock price S_t as following a Geometric Brownian motion. This is modelled by the stochastic differential equation (SDE):
   $$dS_t = \mu S_t dt + \sigma S_t dW_t.$$
   Where:
   - S_t is the stock price,
   - μ is the drift term, which represents the expected return of the stock, $\mu \in \mathbb{R}$
   - σ is the volatility, the measure of uncertainty or risk (mentioned in pervious BM derivation), $\sigma > 0$
   - dW_t is an increment of a standard Wiener process which introduces randomness.
   - $S_0 > 0$, thus ensuring that $S_t > 0$ almost surely (Fixing one of the limitations of Bachelier's model)
   - It follows a log normal distribution (The natural logarithm of the relative price $ln(S_t/S_0)$, follows a normal distribution, implying that the stock price itself is normally distributed), $ln(S_t/S_0) \sim N((\mu - \frac{\sigma^2}{2}), \sigma^2 t)$.

   This model captures both the predictable growth (via the drift) and the randomness (via the volatility and Weiner process) soon to be gone XD.

2. **Ito's Lemma**
   Since the price $V(S, t)$ of a derivative/option depends on both the stock price S and time t, we need a way to describe its instantaneous change. This is where Ito's lemma comes in.
   Let X_t be an Ito process:
   $$dX_t = a_t dt + b_t dWt,$$
   Thus:
   $$dX_t = a(X_t, t)dt + b(X_t, t)dWt,$$

Let $f(X_t, t) \in C^{2,1}\mathbb{R}[0, \infty)$. For a twice-differentiable function $f(X_t, t)$, the differential df is:

$$df = (\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial X^2})dt + b\frac{\partial f}{\partial X}dW_t$$

Applying it to option pricing:
Let $V(s, t)$ be the option price. Since S_t follows GBM:

$$dV = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt + \sigma S\frac{\partial V}{\partial S}dW_t$$

where:

- The term $\frac{\partial V}{\partial t}$ captures how the option's value changes with time
- The term $\mu S\frac{\partial V}{\partial S}$ reflects the effect of the average rate of change of S on V
- The term $\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$ represents the affact of the convexity of V accounting for the random fluctuations in S.
- The term dW_t represents the random shock affecting V due to the movement in S.

This is derived by substituting $a = \mu S$ and $b = \sigma S$ into Ito's Lemma. Ito's Lemma provides the full differential of V, considering both the drift and volatility of the underlying asset.

3. **Constructing a Risk free Portfolio**
In order to eliminate the stochastic component dW_t, construct a self financing portfolio:

- Short 1 derivative: Value $V(S, t)$.
- Long Δ shares: Value ΔS.
Thus the portfolio Value is given by:

$$\Pi = -V + \Delta S,$$

The change in portfolio value is:

$$d\Pi = -dV + \Delta dS.$$

Substituting dV and dS (from Ito's Lemma):

$$d\Pi = [(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \sigma S \frac{\partial V}{\partial S} dW_t] + \Delta(\mu S dt + \sigma S dW_t).$$

Simplifying, We get:

$$d\Pi = (-\frac{\partial V}{\partial t} - \mu S \frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Delta \mu S)dt + (-\sigma S \frac{\partial V}{\partial S} + \Delta \mu S) dW_t.$$

4. **Delta Hedging:**
   In order to eliminate the dW_t term signifying risk, set:

$$-\sigma S \frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0 \implies \Delta = \frac{\partial V}{\partial S}.$$

   That is so satisfying...
   This strategy is known as delta - hedging. Substituting Δ, the portfolio becomes **risk free**:

$$d\Pi = (-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt.$$

5. **No-Arbitrage Condition**
   A risk free portfolio must grow at the risk free rate r:

$$d\Pi = r\Pi \, dt.$$

   Substitute $\Pi = -V + \frac{\partial V}{\partial S} dt.$:

$$(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt. = r(-V + \frac{\partial V}{\partial S} S)dt.$$

   Then we simplify to get the **Black-Scholes-Merton PDE**:

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

6. **Risk-Neutral Valuation:**
   Under risk-neutral measure $\mathbb{Q}$(by Girsanov's theorem), the drift μ is replaced by r and the discounted stock price $e^{-st} S_t$ becomes a martingale (a sequence of random variables where the expected value of the

next value, given all past values, is equal to the current value). The Geometric Brownian Motion SDE transforms to:

$$dS_t = rS_t\,dt \,+\, \sigma S_t\,dW_t^{\mathbb{Q}},$$

where $W_t^{\mathbb{Q}} = W_t + \frac{\mu - r}{\sigma}t$ is a $\mathbb{Q}$-Brownian motion. **Girsanov's Theorem**

- Statement: if $\mathbb{Q}$ is defined by Radan-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp(-\theta W_t - \frac{1}{2}\theta^2 t), \quad \text{where} \quad \theta = \frac{\mu - r}{\sigma}.$$

which enables us to remove μ, the drift, replacing it with r making the discounted process a martingale.

7. **Solving the Black Scholes PDE**
Risk-neutral valuation then allows pricing derivatives as the discounted expected value of their payoff under $\mathbb{Q}$, linking the PDE to probabilistic expectation via the Feynman Kac formula. For Example, the price $C(S,t)$ of an European call option with strike K, maturity T and payoff $max(S_T - k, 0)$ is:

$$C(S,t) = e^{-r(T-t)}E^{\mathbb{Q}}[max(S_T - k, 0) \,|\, S_T = S]$$

steps:

- Log transformation: Let $x = lnS$. The PDE becomes:

$$\frac{\partial V}{\partial t} + (r - \frac{\sigma^2}{2})\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0.$$

- Reduction to Heat Equation:
Define $\tau = T - t, u(x,\tau) = e^{\tau\,t}V(e^x, T - \tau)$. The PDE simplifies to: Define $\tau = T - t, u(x,\tau) = e^{\tau\,t}$

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}.$$

- Integrate with Transition Density:
  The solution under $\mathbb{Q}$ is:

  $$V(S,t) = e^{-r(T-t)} \int_0^\infty max(S_T - K, 0) f(S_T \mid S_t) dS_T,$$

  while $f(S_T \mid S_t)$ is the log-normal density of S_T under Q:

  $$\ln S_T \sim N(\ln S + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)).$$

- Compute the integral:
  The integral splits into two terms:

  $$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

  where $N(\cdot)$ is the standard normal CDF, and :

  $$d_1 = \frac{\ln(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}, \; d_2 = d_1 - \sigma\sqrt{(T-t)}.$$

  Here, $N(d_1)$ and $N(d_2)$ arise from integrating the log-normal density over $S_T > K$.

4.3 Stochastic Calculus

Stochastic Calculus is the backbone of modern mathematical finance, enabling us to model and analyse systems driven by randomness. Unlike classical Calculus which deals with deterministic functions, Stochastic Calculus extends these concepts to include random processes like Brownian Motion. It provides the tools to describe how unpredictable events evolve over time; key to understanding financial markets, where uncertainty reigns supreme. This field emerged as a response to the need for mathematical rigour in handling noise or unpredictable systems. We have already covered fundamental theorems in stochastic calculus such as Brownian Motion, Geometric Brownian Motion, Ito's Lemma, Girsanov's Theorem, ect.

4.4 Kiyoshi Ito(1940s)

Kiyoshi Ito developed Ito's Lemma which is foundational to continuos time finance (It has been previously derived as a prerequisite for the derivation of the Black Scholes Model). Ito's lemma essentially Mathematically captures all the influences of the current stock price (S), time (t) and the random stock price movements dS on the value of an option $V(S,t)$ and is essential in understanding how prices of derivatives evolve when the underlying asset follows a random process.

What does Ito's Lemma Allow us to do?

- Chain Rule for Stochastic Processes: Ito's lemma us the Stochastic counterpart to chain rule. When you have a function $V(S,t)$ that depends on a stochastic process S_t and time t, the lemma tells you exactly how V evolves with time.

- Derivation of Differential Equations: By applying Ito's lemma, we can transform an SDE for the underlying asset into an SDE for any sufficiently smooth function of the asset. We have done an example of this application for Ito's lemma above.

- Quantifying the Effect of Volatility: The lemma introduces an additional term specifically the $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ term when applied to a function of GBM, captures the effect of volatility on the curvature of the function. Thus showing how randomness (via volatility) interacts non-linearly with the function being studied.

- Modelling Under Uncertainty: It provides the necessary mathematical framework to model financial instruments under uncertainty, enabling risk management, hedging strategies (like delta hedging), and pricing of complicated derivative products.

Limitations of Ito's Lemma:

- Ito's Lemma requires that the function $V(S,t)$ be sufficiently smooth; it must be twice continuously differentiable with respect to S and once with respect to t. Functions that lack this smoothness can't be handles by Ito's lemma (We covered this as one of the assumptions for Ito's Lemma).

- The standard form of Ito's Lemma works only for processes that have continuous paths(typically, Ito's processes). If a process includes jumps or discontinuities, as in some jump-diffusion models), then the basic form must be modified or replaced by a jump version of Ito's Lemma.

- While Ito's Lemma accurately describes local changes in the process, its use in approximations or numerical schemes needs care because error terms might accumulate over time if not properly managed.
- Requires an Ito Process Framework: Ito's Lemma is formulated for semi-martingales(a broad class of stochastic processes). If the process doesn't fall into this category, alternate tools or extensions are needed.

Overall, Ito's Lemma is a powerful tool in Stochastic Calculus as it enables us to derive the dynamics for complex functions of random processes (like option prices based on underlying asset prices) and is fundamental to risk management and derivative pricing. However, as we have established, its effectiveness is closely tied to its assumptions; particularly regarding the smoothness and continuity and care must be taken when dealing with processes that fall outside these constraints.

4.5 Gisiro Maruyama (1955)

Gisiro Maruyama's contributions in 1955 significantly broadened the scope of stochastic calculus by extending it from one-dimensional processes to multidimensional ones. In practical terms, here's what that means and why it matters:

- Before Maruyama: Classical Ito calculus was originally developed for one-dimensional processes. For tracking the random movement of a single asset, like the price of one stock; it might follow a path influenced by a single source of randomness(This is one dimensional Brownian motion).
- In comes the Multidimensional Process: In the real world, however, many systems are inherently multidimensional for example an portfolio that depend on several assets, each with its own unique source of randomness, or an economic model influenced by multiple risk factory. These processes are not independent and can be correlated and interact in complex ways.
- Maruyama's Contribution: Gisiro Maruyama extended the mathematical tools of stochastic calculus to handle multiple interacting sources of randomness simultaneously. He generalized concepts such as stochastic

integrals and Ito's Lemma so that they work with vector-valued functions and multi-dimensional Brownian Motions.
In one dimension, we consider an Ito process for a scalar variable S_t satisfying

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

and Itô's Lemma tells us how a function $f(S_t, t)$ evolves—accounting for the drift, volatility, and curvature.

Maruyama extended these ideas to multidimensional processes. Specifically, let $X_t \in \mathbb{R}^n$ satisfy

$$dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t,$$

where:

- $\mu : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is the drift,
- $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ is the diffusion matrix,
- $W_t \in \mathbb{R}^d$ is a standard d-dimensional Brownian motion.

For a function $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ that is twice continuously differentiable in x and once in t, the multidimensional Ito formula is

$$df(t, X_t) \frac{\partial f}{\partial t}(t, X_t) \, dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t) \, dX_t^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) \, dX_t^i \, dX_t^j.$$

Using the rules

$$dW_t^k \, dW_t^\ell = \delta_{k\ell} \, dt, \quad dW_t^k \, dt = 0, \quad dt^2 = 0,$$

and substituting for dX_t, we obtain the compact form:

$$df(t, X_t) \frac{\partial f}{\partial t}(t, X_t) \, dt + \nabla_x f(t, X_t)^\top \mu(t, X_t) \, dt$$

$$
\begin{aligned}
&+\nabla_x f(t, X_t)^\top \sigma(t, X_t)\, dW_t \\
&+\frac{1}{2} tr\Big(\sigma(t, X_t)\sigma(t, X_t)^\top \nabla_x^2 f(t, X_t)\Big)\, dt
\end{aligned}
$$

where $\nabla_x f$ is the gradient, $\nabla_x^2 f$ the Hessian, and $tr(\cdot)$ denotes the trace.

- This expansion allowed for modelling interactions as incorporating the cross-dependence between different sources of risk or randomness.
- This also allowed for robust simulations such as developing numerical methods such as the Eular-Maruyama method to approximate solutions for systems of stochastic differential equations (SDEs) in higher dimensions.
- This also allowed for enhanced financial modelling: Creating models that more accurately reflect the real world dynamics of markets, where multiple factors (like interest rates, asset prices, and volatility) interact.

5 The Computational Revolution (1950s - 2000s

I believe that it is important to dedicate an entire section to the Revolution of computational methods in the field of Financial Mathematics.The Computational Revolution transformed mathematical finance by shifting many once theoretical models into practical, real-world applications, all thanks to dramatic advances in computing power and techniques. In this section, we explore how early computers and pioneering numerical methods reshaped the discipline. From early computing and algorithmic advancements to the ultra-fast trading systems of today, demonstrating how the computational era has revolutionized the ability to model, simulate, and ultimately profit from the complexities of modern financial markets.

5.1 Monte Carlo Simulations

John von Neumann advocates for Monte Carlo methods in solving partial differential equations.During the 1950s to 1970s, computers were just beginning to be used to tackle complex mathematical problems. John von Neumann, a visionary in the field, recognized that many PDEs, which arise naturally in physics and finance might be solved not by traditional discretization methods but via simulation. Monte Carlo methods, which rely on repeated random sampling, became a key tool in this endeavour.

The key idea behind Monte-Carlo Sims is that PDEs, especially those that are high dimensional or arise in stochastic settings, can be related to the Expected values of some random process. The Monte-Carlo Sims approximate these expectations by simulating many sample paths of the process and averaging the results. This simulation-based approach avoids the problems of dimensionality that plagues grid-based methods.

Let us explore some Example uses of Monte Carlo Sims in Financial Mathematics:
Example 1: Solving the Black-Scholes PDE via Monte Carlo Simulation
The Black-Scholes equation for pricing a European call option is given by:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

subject to the terminal condition $V(S,t) = \max(S - K, 0)$.

By the Feynman–Kac theorem, the solution of this PDE can be represented as the expected discounted payoff under a risk-neutral measure:

$$V(S,t) = E[e^{-rT} \max(S_T - K, 0).$$

where the stock price S_T is modelled by a Geometric Brownian motion:

$$S_T = S_0 \exp[(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z],$$

with $Z \sim N(0,1)$.
Monte Carlo Application:

1. Simulate Paths: Generate a large number of sample paths for S_T using the above formula.

2. Compute the payoffs: For each path, calculate the option payoff $\max(S_T - K, 0)$

3. Discount and Average: Multiply each payoff by e^{-rT} (to discount the present value) and take the average over all simulations. This average approximates the option price.

Example 2: Consider the classic heat (diffusion) equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with suitable initial boundary conditions. This solution of this equation can be represented as an Integral involving the fundamental solution (heat kernel), which is itself related to the probability density of a Brownian motion (or random walk).

Monte Carlo Application:

1. Random Walk Simulation: Instead of discretizing the spatial domain finely, you generate many random walk paths that mimic the diffusion process.

2. Path Averaging: The value of $u(x, t)$ at a given point is then approximated as the average of the contributions from each random path, thereby "solving" the PDE without explicitly constructing a numerical grid.

Benefits and Scalability

- Scalability: Monte Carlo methods scale much more gracefully with the dimension of the problem, making them particularly appealing for high-dimensional PDEs or when the randomness is complex

- Flexibility: This approach can handle complicated boundary conditions and non-linearities that are difficult to tackle via traditional methods

- Von Neumann's advocacy of Monte Carlo methods laid the groundwork for the computational finance revolution. Early implementations enabled the computation of option prices, risk measures, and the behaviour of physical systems, thus proving essential in both theoretical research and practical applications.

6 AI/Quantum Era (2000's - Present)

Over the past few decades, there has been a rapid surge in the use of AI (Artificial Intelligence) across various fields, including process automation and enabling computers to extract meaningful information from data. Computer Vision (CV) models empower computers to detect and identify images, while Large Language Models (LLMs) interpret human input and perform a range of tasks. These areas of AI fall under the broader category of Machine Learning.

Machine learning is typically employed to uncover relationships between data or predict outcomes based on a "training dataset." It can also be used to classify information by utilizing algorithms to identify similarities between different pieces of data. Deep learning, a subset of machine learning, focuses on leveraging neural networks to uncover more complex connections within data. This approach underpins the majority of advancements in Mathematical Finance seen today.

6.1 Machine Learning (2000 - 2020)

Neural Network Models revolutionized option pricing by approximating highly non-linear functions that relate Various markets inputs (such as the underlying asset price, volatility and time to maturity) to option premiums. One compelling example comes from research conducted at Stanford University. In their study, researchers built and compared several architectures including Multi-layer Perceptions ($MLPs$) and Long Short-term Memory ($LSTM$) networks to estimate option prices using data that traditionally would be fed into the Black-Scholes model. The MLP variants were fed summary statistics like 20-day historical volatility, while the LSTM was given sequential price data over the same period. Their experimental results showed that the neural network models could capture complexities (e.g., the volatility smile) much more effectively than the standard analytical model, thereby reducing pricing errors considerably.

Another practical example is illustrated by an article on Quant Next, which involved a python implementation of an artificial neural network for option pricing. In that implementation, they set up a supervised learning model where the network was trained on historical option prices (or even synthetic prices obtained from Black-Scholes as a baseline) using a loss metric like mean squared error (MSE) and optimized with algorithms such as Adam. This approach not only offered superior performance compared to traditional numerical methods (like binomial trees or Monte Carlo simulations) but also demonstrated the flexibility of neural networks in adapting to a variety of option types with complex payoff structures

Reinforcement learning (RL) entered the trading arena as a way to directly learn trading strategies by framing the trading process as a sequential decision-making problem. An example of this is the open-source FinRL framework from the AI4Finance Foundation, which is hosted on GitHub. FinRL provides a ready-to-use pipeline: market environments are simulated, and RL agents who using algorithms such as Deep Q-Networks (DQN), Policy Gradient methods, or Actor Critic architectures that learn from interacting with these environments, gradually honing strategies that maximizes accumulating returns. The framework includes a variety of environment setups (from stocks and ETFs to cryptocurrencies), offering practical notebooks that demonstrate everything from training and testing to real time usage. The Oxford-Man Institute has become a focal point for using ML to tackle the complexities of financial markets. Their research spans a wide range of topics, including deep neural networks for option pricing, risk management, and high-frequency trading. For instance, the ELLIS Oxford unit describes projects aimed at building ML systems that can forecast multi-horizon market movements and extract signals from noisy financial data.

A particular project, called "Deep Learning or Quant Finance Strategies" demonstrate how classical momentum strategies can be enhances using RNNs (Recurrent Neural Networks) such as the LSTM networks and gradient boosting methods. This work has advanced the field by effectively removing the need for heuristic trend estimators (some rules based on past market trends used for predicting new market trends).

6.2 Quantum Finance (2020's - Future)

Quantum Finance is the new kid on the block in the field of Quantitative Finance hence it's effectiveness has not been fully established, However, recent research in quantum computing has been promising in targetting the notoriously challenging portfolio optimization problem which is a cornerstone of Quantitative Finance. Traditionally formulated as a quadratic optimization problem (as shown prior), portfolio optimization becomes computationally intensive as the number of assets increases. Quantum algorithms, by contrast, promise significant speed-ups whilst processing this vast amounts of data. Two key strands illustrate these efforts:

1. **Quantum Annealing Approaches:**
   A notable example of this approach is outlined in the paper "A Real World Test of Portfolio Optimization with Quantum Annealing" written in March 2023. In this study, researchers reformulated the portfolio optimization problem as a Quadratic Unconstrained Binary Optimization (QUBO) problem. This is how it works:

   - Firstly, the QUBO problem is formed by reformulation of the specific portfolio optimization problem. The portfolio's objectives (such as maximizing return and controlling the risk via variance) are encoded into this model.
   - Hybrid Quantum Classical Solvers: This work employed D Wave's hybrid solvers, which use a combination of quantum annealing and classical optimization routines. The quantum annealer explores the solution space efficiently, while classical methods help tune the QUBO parameters which is very crucial for capturing constraints such as maintaining the portfolio's variance below a pre-determined threshold.
   - This Method was tested with data from real world financial systems in collaboration with financial institutions such as Raiffeisen Bank International, the experiment achieved results very similar to the classical exact methods whilst also hinting at the potential for quantum speed-ups in larger and more complex portfolios.

   These findings indicate that quantum annealing can effectively navigate the complex energy landscapes of financial optimization problems thus offering a promising path for real-time portfolio management in the near future. This promise grows stronger as computational capabilities continue to evolve.

2. **Gate Based Quantum Algorithms**
Another exiting and promising development comes from research found in even more recent publications such as those found in a springer article from 2024 by Rebontrost, Lloyd and colleagues. This work focuses on:

   - This work focuses on speeding up Linear Algebra using quantum setups (i.e leveraging quantum routines), similar to the HHL (Harrow Hassidim Lloyd) algorithm; researchers have designed algorithms that can can essentially process market data and compute the optimal risk to return trade-off curve in time scaling as poly(log(N)) rather than poly(N) where N is the number of assets.
   - Sampling the Optimal Portfolio: The algorithm produces a quantum state that encodes information about the optimal portfolio, from which one can sample to extract implementable portfolio configurations. This approach shows how quantum computing can potentially transform portfolio optimization by tackling problems that are intractable on classical computers.

   While Gate based Quantum algorithms are still largely theoretical and contingent on the development of fault tolerant quantum hardware, they set the stage for long-term breakthroughs in computational finance.

Broader Implications:
The promise of quantum algorithms in portfolio optimization is just one facet of the emerging field of quantum finance. Researchers are also exploring integrating quantum methods in order to mitigate risk by building a hybrid model where quantum enhanced optimization techniques directly feed into risk management systems. Researchers are also exploring some cross disciplinary applications of quantum algorithms in different financial domains such as quantum blockchain security or fraud detection and focusing on how to scale these algorithms to handle an increased number of assets and more realistic market conditions, as well as addressing hardware limitations and error correction challenges.

Thanks

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I will continue to add to this document including some code snippets and images for the Monte-Carlo Simulations and include even more derivations that I find interesting. This was pretty fun and I will definitely be doing more projects similar to this one in the near future. Updates to this document and future projects can be found in my LinkedIn @`www.linkedin.com/in/daniel-igwelaezoh-6762511b6`

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