Hebbian Theory: Cells that fire together wire together.

1. Abstract

This paper explores the Hebbian theory of learning, a foundational concept in neuroscience which suggests that "neurons that fire together, wire together." Starting with a biological perspective, we examine how neurons form connections through repeated activity and how this mechanism supports human memory and learning. We then translate these ideas into mathematical models, including the classic Hebbian rule and its refinements such as Oja's rule, the covariance rule, and the Bienenstock-Cooper-Munro (BCM) model. These models highlight how synaptic plasticity can be regulated to prevent runaway growth and support stable learning. By combining insights from neurobiology and mathematics, this paper aims to provide a clear and accessible understanding of how the brain learns, adapts, and forms long-term knowledge structures.

2. Introduction

Everyone had these nights. One day before the exam and you have to cram hundrets of pages in a few hours. The feeling of wanting to give up is overwhelming. But how does the human brain even learn? How is it possible that someone have such brod knowledge and some can not seem to remember the simplest concepts?

In the year 1949, Donald Hebb wrote down his cell assembly theory in the book "The organization of behaviour

"Let us assume that the persistence or repetition of a reverberatory activity (or "trace") tends to induce lasting cellular changes that add to its stability. ... When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that *A*'s efficiency, as one of the cells firing *B*, is increased."

This may seem now a bit confusing and a lot of words, but in a bit, I will show you how mathematics can us help understand the human brain and how does one properly learn and create connections.

3. Neurology

3.1. The blueprint of our brain

So we have our brain – some have more of it, some less of it. The brain is made up of billions of cells called neurons and these neurons are connected to each other through synapses, which are small gaps that allow neurons to communicate. The brain has different regions that handle various functions, like movement, thinking, memory, and emotions. These regions work together to process and store information.

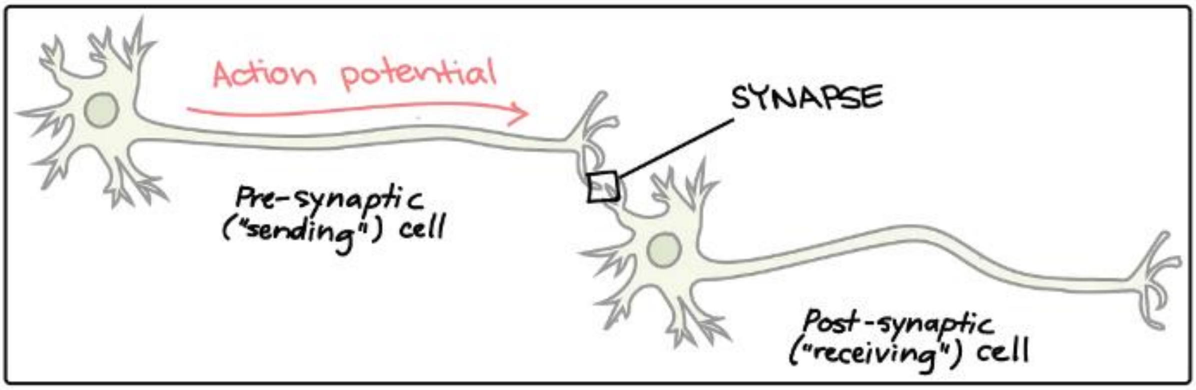


Figure 1 Neurons in action

3.2. Human Memory

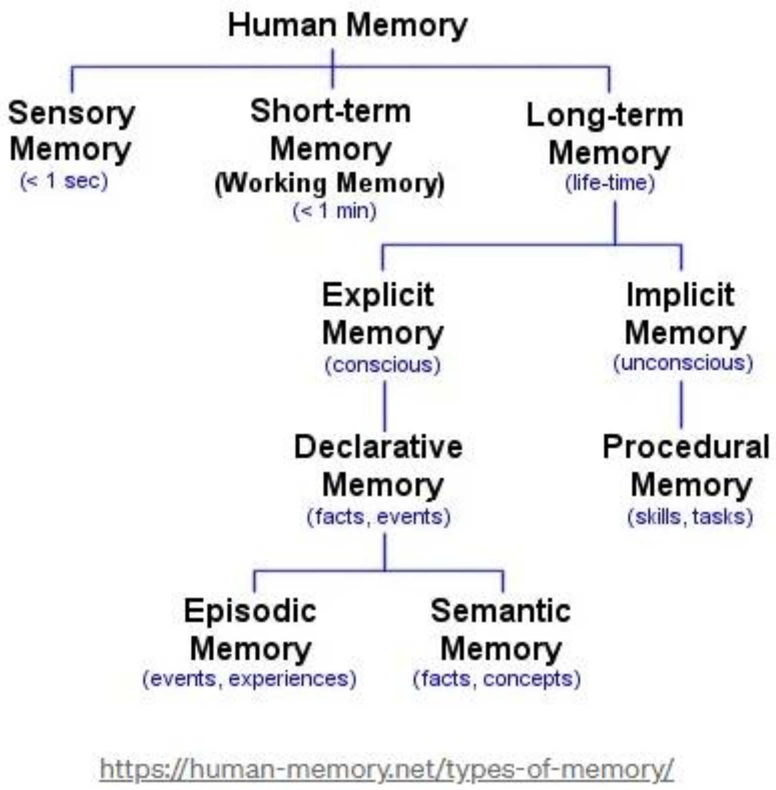


Figure 2 Human Memory

There are also differences between human memory. There is for example the Sort-term Memory which just hold temporarily information. For studying we relied on the Declarative memory (Involves facts and events, like the name of the current prime minister) and the Procedural memory (Responsible for skills and actions, like riding a bike or playing an instrument.)

3.3. Synaptic Plasticity

Our brain always changes! Isn't that fascinating

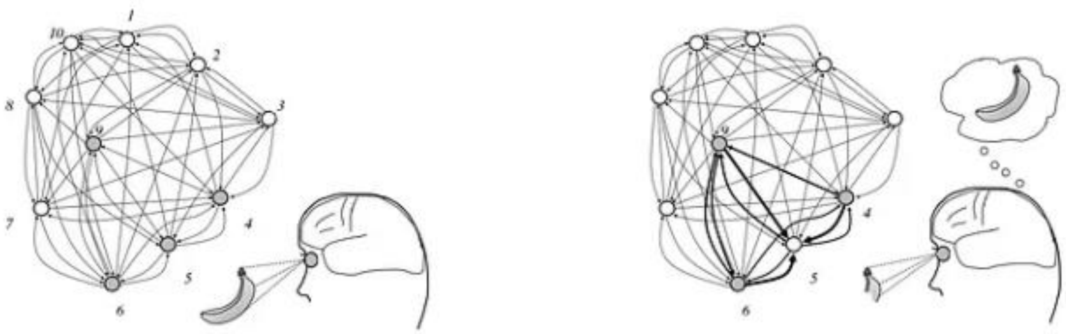
And Synaptic plasticity refers to the ability of synapses (the connections between neurons) to strengthen or weaken over time. This change in the synapse is how learning happens in the brain. When neurons communicate frequently, the synapse connecting them becomes stronger, making it easier for them to communicate again in the future.

3.4. What is even learning?

Learning in this context means adjusting the "weights" or strengths of the connections between neurons so that the network can work better. When two neurons repeatedly fire together, their connection strengthens, improving the brain’s ability to process information. This is the essence of Hebbian learning.

3.5. Learning Mathematical Topics

Let’s say you're learning about Hebbian Learning for the first time. Initially, your brain doesn’t have any connections related to the concept. As you read the definition—how neurons that fire together wire together—your neurons for "learning," "neuron activity," and "connections" start firing. These neurons activate areas of your brain associated with other learning concepts, like memory and patterns. The more you read and understand how Hebbian Learning works, the stronger the connections between those neurons become. Over time, when you encounter a new related topic, like synaptic plasticity or neural networks, your brain quickly activates the learned connections and retrieves the Hebbian Learning concept more easily.



Gerstner (2011): Representation of Hebbian Learning in Human

Figure 3 Connections starting to from!

You can imagine it like in the picture above – just without the banana.

4. Mathematical Foundations

4.1. What did Hebb say?

So, what did Hebb actually say? The core idea is simple: *"Neurons that fire together, wire together."* That means if two neurons are active at the same time, the synaptic connection between them becomes stronger.

However, this basic principle is a bit too vague for practical mathematical modeling. It works fine as a biological hypothesis, but in math, we need something more precise and structured.

4.2. Mathematical Formulation of Hebb's Rule

Let's take a closer look at Hebb's postulates. Two main aspects are essential here: **locality** and **joint activity**.

- **Locality** means that changes in synaptic strength depend only on local information—that is, what's available at the synapse itself.
- **Joint activity** means that both the presynaptic and postsynaptic neurons are active at the same time.

$$\frac{\mathrm{d}}{\mathrm{d}\ t} w_{ij} = F\left(w_{ij}; \nu_i, \nu_j\right) .$$

Figure 4 General Formula

This represents the *rate of change* of the synaptic strength wji, the connection weight from neuron j to neuron i.

As mentioned earlier, if one of the neurons isn't active, nothing happens. That's the essence of joint activity.

4.3. Hebb's Rule in Mathematical Terms

$$\frac{\mathrm{d}}{\mathrm{d}\ t} w_{ij} = c_{11}^{\mathrm{corr}} \quad \nu_i \, \nu_j .$$

Figure 5 Hebb's Rule in Mathematical Terms

- *wij:* Strength of the connection from neuron jjj to neuron iii
- *vi, vj:* Activities (firing rates) of neurons iii and jjj
- *c:* A positive learning rate (constant)

This equation can be seen as the simplest version of a Taylor expansion, using a linear approximation to describe synaptic change. Importantly, it only allows **positive growth** (since c>0), reflecting Hebb's original idea: connections strengthen with joint activity.

But there is a problem!

This version is quite unrealistic—like using a spoon to dig a tunnel. Synapses don't just grow infinitely strong. So, two major refinements have been introduced:

Hard Bound: Learning stops when the weight reaches a maximum, i.e., wij≥wmax

Soft Bound: As wij increases, the learning rate decreases. The stronger the connection, the harder it is to grow further.

$$c_{11}^{\text{corr}} \quad (w_{ij}) = \gamma_2 \left(w^{\max} - w_{ij}\right)^{\beta} ,$$

Figure 6 Soft Bound

4.3.1. Another Problem

Yeah!

There should also be a possibility for a depression!

This is possible by through simply adding $-\gamma_0 w_{ij}$. Now this formula could describe the growth and decline of synapses.

$$\frac{\mathrm{d}}{\mathrm{d} \quad t} w_{ij} = \gamma_2 \; (1 - w_{ij}) \; \nu_i \, \nu_j - \gamma_0 \, w_{ij} \, ,$$

Figure 7 Now are Depression possible!

4.4. Specializations of Hebbs' Rule

You make ask yourself what I mean with speculations. You can see, the general Hebbian rule is just the core and there are a lot of different models which are more realistic or detailed. And now I will also tell you a bit of the specializations and what makes them special.

4.4.1. Covariance rule

Neurons should strengthen their connection only when their activities are *correlated*, not just coincidentally high.

- If both neurons are firing above or below their average rates (positive covariance), the synapse is strengthened.
- If their firing rates move in opposite directions, the synapse is weakened.

To apply this, we assume that the average firing rates ν_i and ν_j are constant over time. We then adjust the learning rule to consider fluctuations around these means.

$$\frac{\mathrm{d}}{\mathrm{d} \quad \mathrm{t}} w_{ij} = \gamma \; (\nu_i - \langle \nu_i \rangle) \; (\nu_j - \langle \nu_j \rangle) \; ,$$

Figure 8 Covariance rule

4.4.2. Oja's rule

Oja's rule prevents unbounded growth by **normalizing** synaptic weights, ensuring stability and convergence.

$$\frac{d}{d\,t} w_{ij} = \gamma \left[\nu_i \, \nu_j - w_{ij} \, \nu_i^2\right]$$

Figure 9 Oja's rule

Where:

- The first term $\gamma \cdot \nu i \cdot \nu j$ is the classic Hebbian term.
- The second term $-\gamma \cdot wij \cdot \nu i^2$ ensures that the weights don't grow uncontrollably.

This introduces **competition** between synapses. If one synapse strengthens, others must weaken to maintain a constant overall norm:

$$\sum_j w_{ij}^2 = 1$$

Figure 10 COMPETITION in mathematic terms

You may ask yourself why this works!

The answer is simple. Oja's rule maintains Hebbian learning while solving its runaway growth problem. By damping the increase with a quadratic term, the system reaches a steady state and only relevant patterns are reinforced. Meanwhile there is the competition, when one weight increases, others must decrease. This reflects biological reality: neurons can only effectively maintain a limited number of strong synapses

4.4.3. Bienenstock-Cooper-Munro rule

Let's now take a look at a more advanced learning rule: the **Bienenstock-Cooper-Munro (BCM) rule.** This rule introduces a nonlinear component to synaptic plasticity and can be understood as a generalization of Hebb's principle.

At its core, the BCM rule proposes that changes in synaptic strength are not simply a matter of joint activity, but depend on how the postsynaptic neuron's activity compares to a *dynamic threshold* $\nu 0$.

You may ask yourself what a threshold is? The BCM rule says we don't compare activity to a fixed level. Instead, we use a **moving target**, called a **threshold.**

This threshold is like a student's **personal best.** If they do better than their own usual level, they get rewarded. If they do worse, they might get a little penalty. This way, learning is **personalized**—it depends on how active a neuron *normally* is.

The key equation is:

$$\frac{d}{d\ t} w_{ij} = \phi\ (\nu_i - \nu_\theta)\ \nu_j$$

Figure 11 BCM Rule

This dynamic behavior is crucial: if the postsynaptic activity is **moderate** (i.e., below the threshold), the synapse **weakens**. If the activity is **strong** (above the threshold), the synapse **strengthens—**but only at the synapses where presynaptic activity is present. In this way, the rule enforces **activity-dependent selectivity**.

A commonly used form of this rule is:

$$\frac{d}{d\ t} w_{ij} = \eta \nu_i\ (\nu_i - \nu_\theta)\ \nu_j = c_{21}\nu_i^2\nu_j - c_{11}^{\text{corr}}\ \nu_i\nu_j$$

Figure 12 BCM Rule interpreted as a Taylor expansion

This can be interpreted as a **Taylor expansion**, where:

- $c21=\eta$ is the learning rate for the strengthening term.
- $corr,11=-\eta\nu\theta$ represents a correction term that introduces depression when activity is too high.

An important feature of this rule is that for **stationary input**, the postsynaptic activity νi tends toward a **fixed point** at $\nu\theta$. However, this fixed point is **unstable—**meaning that without further control, activity could either grow uncontrollably or drop to zero. To prevent this, the threshold $\nu\theta$ must adapt based on the neuron's recent average activity, allowing for stable learning over time.

The BCM rule has practical implications: it explains how **neurons can become selective** for specific patterns of input. This selectivity is essential in **developing receptive fields**, for instance in the visual cortex, where neurons learn to respond more strongly to specific orientations or movements.

In short, the BCM rule captures the idea that synaptic change depends not just on correlation, but on how surprising or significant the activity is compared to a dynamic baseline—adding a rich layer of biological realism to models of learning.

5. Results and Discussion

So what's the result. I wrote and wrote about different formulas but what now?

There are a lot of ways to describe Hebb's Learning, also a lot that I didn't even mention like, Pair-based STDP *(Direct extension of Hebb: Strengthens synapses only if the presynaptic spike comes shortly before the postsynaptic one – otherwise, it weakens. This is Hebbian plasticity with timing),* Triplet-STDP *(Another extension: Considers multiple spikes and frequencies, still based on Hebb but in more detail.)* and Voltagedependent STDP *(Hebbian learning + additional biological factors, like membrane potential – still based on Hebb but more realistically modeled.).*

Which formula suits you depends on the situation. For newbies would be the general formula enough to understand the principle: Cells that fire together wire together. But for multifaced situation, where one wants to go into detail , the Bienenstock-Cooper-Munro rule, would do that perfect, because this rule is like a "smart" version of the basic idea. It says that neurons can only strengthen their connections if they fire together often enough, but it also has a "threshold" or limit, so the connections don't get too strong or unstable. This helps to make sure that the brain stays balanced and doesn't go overboard in strengthening connections.

6. Conclusion

In conclusion, Hebbian learning offers a powerful and intuitive explanation of how our brains adapt and form memories: "Cells that fire together wire together."

So, returning to the question from the beginning of this essay—what about cramming the night before an exam? It's certainly rough if you're unfamiliar with the topic, but not impossible. The key is to engage as many neurons as possible by making the subject interesting and meaningful to you. The more actively and emotionally you connect with the material, the stronger the neural pathways become, making learning more effective—even under pressure.

7. References

1) https://www.verywellmind.com/what-is-learning-2795332
2) https://www.researchgate.net/publication/5227053_A_Mathematical_Analysis_of_the_Effects_of_Hebbian_Learning_Rules_on_the_Dynamics_and_Structure_of_Discrete-Time_Random_Recurrent_Neural_Networks
3) https://neuronaldynamics.epfl.ch/online/Ch19.S2.html
4) https://becominghuman.ai/hebbian-learning-and-plasticity-i-the-maths-behind-how-your-brain-learns-and-unlearn-3cfa98821637
5) https://thedecisionlab.com/reference-guide/neuroscience/hebbian-learning
6) https://en.wikipedia.org/wiki/Hebbian_theory
7) https://becominghuman.ai/hebbian-learning-and-plasticity-i-the-maths-behind-how-your-brain-learns-and-unlearn-3cfa98821637
8) https://neuronaldynamics.epfl.ch/online/Ch19.html