Making a Biased Coin Fair: A Lesson in Probability

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1 Introduction

It is the Cricket World Cup Final and spirits are at an all time high. You have been appointed to be the umpire for the match. Well known for your impartiality even among all your contemporaries back at the National Association of Fair Umpires, you were the ideal choice for the role. You also have the responsibility of tossing the coin for the match to decide who will be batting first. The coin to be used for the toss is a specially minted one, to ensure that the toss will be fair.

An hour before the game starts, just when you were near the room guarding the special coin, you notice a suspicious man in black leaving the room. You enter the room as fast as you can. The coin must have been tampered with but you have no way of knowing. With the fairness of the match and the outcome of the game on the line, you must use your expertise as the fairest umpire of all time to solve this issue.

2 Solving the Problem

The problem at hand is that we have a coin such that $\mathbb{P}(\text{Heads}) = p$, where p is unknown. We need to make a fair toss using this coin so that $\mathbb{P}(\text{Heads}) = 0.5$.

You don't know whether the new coin is biased towards heads or tails. You can't use a single toss to make a fair toss but you can use multiple tosses. This is where your Bachelors' Degree in Mathematics comes to the rescue.

Figure 1: Sides of a Coin

Suppose for this defective coin,

$$\mathbb{P}(\text{getting H}) = p$$
$$\mathbb{P}(\text{getting T}) = 1 - p$$

where $0 < p < 1$.
Therefore, considering independent tosses (Probabilities of independent events multiply together),

$$\mathbb{P}(\text{getting HH}) = p^2$$
$$\mathbb{P}(\text{getting HT}) = p(1 - p)$$
$$\mathbb{P}(\text{getting TH}) = (1 - p)p$$
$$\mathbb{P}(\text{getting TT}) = (1 - p)^2$$

and so on ....

Notice that if we can get an event A such that $\mathbb{P}(A) = 0.5$ we are done. But is there any way we can get rid of the parameter p? Seems like an impossible task. That's why we will use a clever idea... something called conditional probability.

3 Conditional Probability

Conditional probability (read as probability of A given B) is defined as:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \qquad , \mathbb{P}(B) \neq 0$$

This probability is often viewed as updating one's view after getting some additional information. On one side we have Probability of the event A and on the other side we have Probability of the event A *given* the information that the event B is true.

Note that the formula captures this idea. We are restricting ourselves to the outcomes in event B and the event of interest is the outcomes in A also lying in B and we take the ratio of the two to denote 'Event divided by Total Outcomes'.

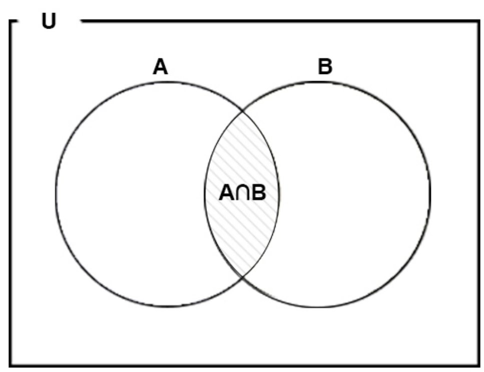

Figure 2: $A \cap B$

But the main thing to notice is that the conditional probability is a ratio of probability of 2 events. So, if we can get events A and B such that $\mathbb{P}(B) = 2 \times \mathbb{P}(A \cap B)$, we are done (assuming the probabilities are non-zero). And with that the answer is right in our face.

4 The Answer

If you have noticed,

$$\mathbb{P}(\text{getting HT}) = \mathbb{P}(\text{getting TH}) = (1-p)p$$

So, if we take
B= getting TH or HT
A= getting TH

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(\text{getting HT}) + \mathbb{P}(\text{getting TH}) \\ &= 2p(1-p)\end{aligned}$$

[Probability of disjoint events add up.]
Therefore,

$$\begin{aligned}\mathbb{P}(TH|\text{HT or TH}) &= \frac{\mathbb{P}(TH)}{\mathbb{P}(\text{HT or TH})} \\ &= \frac{p(1-p)}{2p(1-p)} \\ &= 1/2\end{aligned}$$

[Note that here we need $0 < p < 1$ to prevent division by 0!!]
So with that how do you get a fair toss.

1. You can toss the coin twice. If we get HH or TT, we reject the toss and toss again.

2. We keep tossing until we get HT or TH (this ensures we are in the $\mathbb{P}(\cdot|B)$ regime.)

3. If we get HT we classify that as a **HEAD** and if we get TH we classify it as **TAILS**.

The earlier calculation proves that we will indeed get a fair toss in this case, **irrespective of the value of p**.

With all these calculations in your head, you explain it to the committee and the audience and execute a fair toss. The game proceeds normally and you retain your title as the fairest umpire. :)

5 Final Remarks

This ingenious idea was proposed by the famous mathematician, John von Neumann. It gives us a simple way of using conditional probability and the use of randomness to get rid of bias. You are free to explore other ways of making the toss fair. Hope you enjoyed the read.

Thank you for your time. :D