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Introduction:

Can you name me 3 football players?

Whenever I ask this question, I usually get the following responses: 'I don't watch football' or I get the responses 'Messi' and 'Ronaldo' most of the time from people that watch the sport and other popular answers include 'Neymar Jr', 'Mbappe', 'Haaland', 'Bellingham', Vinicius Jr' and every now and then I get the names of retired players such as 'Zidane', 'Ronaldinho' and 'Iniesta'.

However, do you notice a pattern? They are all male football players.

When the question is asked only approximately 10% of respondents say the name of female football players, for example 'Sam Kerr', 'Alexia Putellas' or 'Alex Morgan' even though women's football is growing at a faster rate than men's football in terms of viewership, investment and participation as the high popularity of men's football is increasing at a slower rate (as it has been around for longer).

For example, the 2023 FIFA Women's World Cup had 2 billion viewers globally compared to 1.12 billion during the 2019 FIFA Women's World Cup (this is exponential growth as reported by Wikipedia).

In this essay I will be explaining how Mathematics and women's football are intertwined and how there should be equality in terms of financial stability for female footballers as despite having differences compared to the men's game (due to the difference in anatomy) the beautiful game is very similar. I will also be exploring how you can use GCSE Mathematical skills to understand football by developing them further.

The Mathematics of Women's Football: Bridging the Gap in Equality and Understanding the Beautiful Game

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Before we start, to follow the more complex analysis in this essay, it is helpful to understand the mathematical notation used throughout (that you may have not covered in your GCSE studies). Below is a concise summary of the key symbols and their meanings:

- $\therefore$ (Therefore): represents a logical conclusion or deduction
- $\Rightarrow$ (Implies): indicates that one statement logically leads to another
- $\rightarrow$ (Used for steps in working out): helps to clearly show the progression of calculations or transformations in a solution.

How does the data link to GCSE Mathematics?

We can model this exponential increase in popularity and viewership using an exponential model. You may recall looking at the shape of graphs similar to $y = 2^x$ during your GCSE Maths studies, but we can generalise the equation of an exponential equation as $y = ab^x$, $y = ae^{kx}$, $y = a(10)^{bx}$.

Using the data from before we can form the following question:

The viewership of FIFA Women's World Cup grows according to an exponential model. V is the number of viewers after t years. It is known that at the start of the data collection (in 2019) the viewership is 1.12 billion and after 4 years the viewership is 2 billion. Find the equation for this exponential model.

However, before we can tackle this question, we need to look at how prior knowledge that you gained during your GCSE can help you to understand and work with $y = ae^{kx}$, so let's go through a quick summary of logs and exponentials:

Logarithms and Exponentials Summary:

- A relationship which is expressed using powers can also be written in terms of logarithms: $a^n = b$ and $\log_a b = n$ are equivalent.
- The general equation of an exponential function is $y = a^x$ where a is a positive constant and the x-$axis$ is an asymptote.
- There is a value for a between 2 and 3 that has special qualities: the graph of $y = e^x$ and the graph of its first derivative are identical ⇒ $e = 2.71828\ (5\ d.p.)$
- We call $\log_e y$ the natural logarithm which we write as $\ln y$.
- $\ln$ and e are inverses of each other so they cancel each other out: $\ln e^x = x$ and $e^{\ln x} = x$.
- If you $\ln$ one side to get rid of the e , you also have to $\ln$ the other side of the equation.

Using the information above, we can tackle the question:

$V = ae^{tx}$

$\rightarrow (1.12 \times 10^9) = ae^{(0)x}$

$\Rightarrow a = 1.12 \times 10^9$

$\rightarrow V = (1.12 \times 10^9)e^{tx}$

$\rightarrow (2 \times 10^9) = (1.12 \times 10^9)e^{4x}$

$\rightarrow \frac{25}{14} = e^{4x}$

$\rightarrow \ \ln\left(\frac{25}{14}\right) = \ln(e^{4x})$

$\rightarrow \ \ln\left(\frac{25}{14}\right) = 4x$

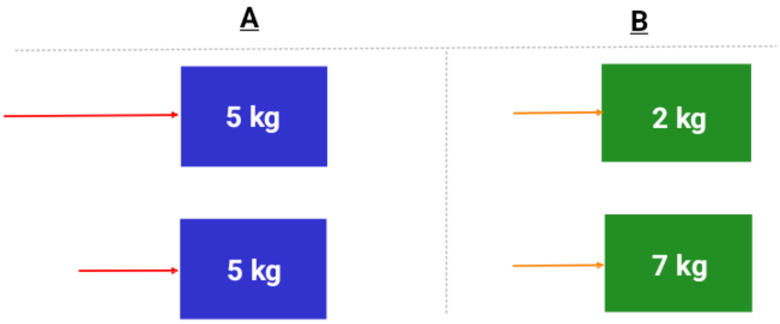
$\Rightarrow x = \frac{\ln\left(\frac{25}{14}\right)}{4} = 0.145\ (3sf)$

$\therefore\ V = (1.12 \times 10^9)e^{(0.145)t}$

Mechanics: $F = ma$

You may remember using the formula $F = ma$ during your GCSE Physics studies as Newton's second law of motion states that 'the acceleration (a) of an object is directly proportional to the net force (F) applied to it and is inversely proportional to its mass (m)'.

Diagram 1:



Just to do a recap: using this law of motion, this means that in A, the 5kg object on the top will have a greater acceleration as it has a greater force for an equal mass as the bottom object, and if we consider B, this means that the object on the top will also have a greater acceleration than the object on the bottom as it has a smaller mass for the the same force being applied as the bottom one.

So you may now be thinking why we are talking about laws of motion in a Mathematics essay if you used $F = ma$ in your Physics GCSE, so let me tell you: in A-Level Mathematics you may study a module on Mechanics in which $F = ma$ plays a significant role, for example the formula is used in: equations of motion, forces and motion, connected particles, inclined planes, momentum and impulse, etc...

But you know be wondering what this has to do with a Mathematics essay based on women's football, so let me answer that too:

It is often (wrongly) thought that women's football is slower paced than men's, as women have a lower acceleration due to a lack of muscles.

However, this is wrong as men and women have similar acceleration as:

Since acceleration is given by $a = \frac{F}{m}$, if both force and mass are lower by similar proportions, the acceleration can remain similar.

- Women typically (not always) have less muscle than men (due muscle fibre differences, lower testosterone, differences in fat percentages), meaning they might produce less force.
- However, they also tend to have lower body mass (due to a smaller skeletal frame, lower muscle density, genetics, etc..), meaning the denominator of $a = \frac{F}{m}$ is also smaller.
- If both force and mass decrease proportionally, the acceleration remains roughly the same.

Differentiation: Displacement, Velocity and Acceleration

You may recall from GCSE that to work out the gradient of a line you have to use $\frac{y_2-y_1}{x_2-x_1}$ and that to find the gradient of a curve at a particular point you have to draw a tangent to that point and then work out the gradient of the tangent.

However, for a curve the gradient varies, and we can no longer have a single value for the gradient; we ideally want an expression in terms of x that gives use the gradient for any value of x.

Differentiation can be used to find the gradient of a function at any point. For a function y we differentiate to get its derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$.

The gradient function at a turning point is equal to 0.

To differentiate x^n:

1. We multiply the coefficient by the power
2. Reduce the power by one

Notation:

- y differentiates to $\frac{\mathrm{d}y}{\mathrm{d}x}$
- $f(x)$ differentiates to $f'(x)$

If we differentiate a function once we get the first derivative, and if we then differentiate the first derivative, we get the second derivative.

How does this link to Maths and women's football?

- Derivative of displacement is velocity.
- Derivative of velocity is acceleration.
- So second derivative of displacement.

Example: A Footballer Running

1. If a player's displacement (the vector quantity that has both magnitude and direction) follows the equation: $s(t) = 2t^3 - 5t^2 + 3t$
2. Differentiate to get velocity: $v(t) = \frac{\mathrm{d}s}{\mathrm{d}t} = 6t^2 - 10t + 3$
3. Differentiate again to get acceleration: $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = 12t - 10$

What this shows us:

- This allows us to analyse how a player speeds up or down over time.
- As time increases, acceleration changes.
- The term $12t$ shows us that acceleration increases linearly over time.

→ At $t = 2$, $a(2) = 12(2) - 10$ ∴ the acceleration at 2 seconds is 14m/s^2, this means that acceleration is positive, so the footballer is speeding up.

Once you have found the displacement function you can work out the velocity and acceleration of any footballer.

Integration: Velocity Function and Displacement

Integration is the reverse process of differentiation.

To integrate we:

1. Add one to the power
2. Divide by the new power

When you integrate the gradient function, you get the original function and when you integrate the original function you find the area bounded between the curve and the x-$axis$.

The area between a curve $y = f(x)$, the x-$axis$ and the lines $x = a$ and $x = b$ is given by: $Area = \int_a^b y\,\mathrm{d}x$ (dx means in respect to x).

Example: Find the area of the finite region between the x-$axis$ and curve with equation $y = (3 - x)(x + 6)$

$\rightarrow y = -x^2 - 3x + 18$

$\Rightarrow \int_{-6}^{3} -x^2 - 3x + 18\ dx$

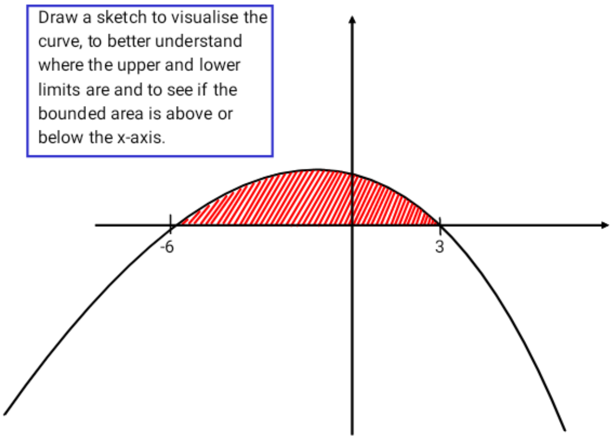
$$\rightarrow \left[-\frac{1}{3}x^3 - \frac{3}{2}x^2 + 18x\right]_{-6}^{3}$$

$$\rightarrow \left[\left(-\frac{1}{3}(3)^3 - \frac{3}{2}(3)^2 + 18(3)\right) - \left(-\frac{1}{3}(-6)^3 - \frac{3}{2}(-6)^2 + 18(-6)\right)\right]$$

$$\rightarrow \left[\frac{63}{2} - (-90)\right]$$

$$\therefore\ \frac{243}{2} units^2$$

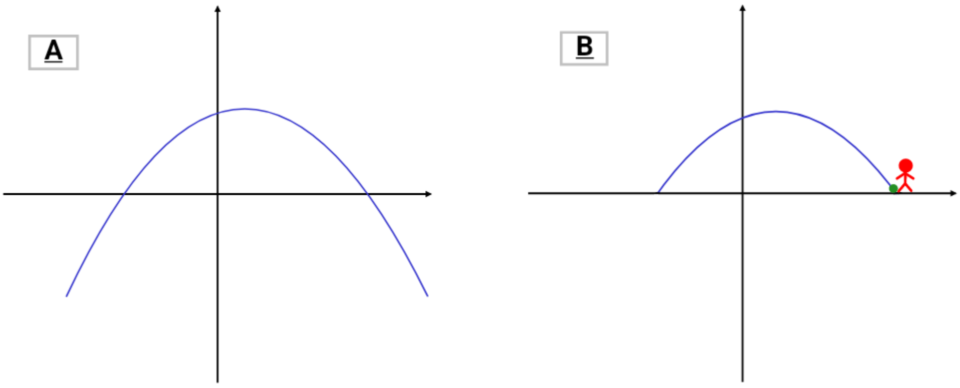
Draw a sketch to visualise the curve, to better understand where the upper and lower limits are and to see if the bounded area is above or below the x-axis.



Therefore, if you integrate a negative quadratic velocity function, you will obtain the displacement (this is the reverse method as the last section on differentiation).

Modelling with Quadratics: Projectile Motion

Have you ever wondered how, when a goalkeeper – like Mary Earps – kicks a football into the air, we can determine how high it goes and how it moves through the air? The answer lies in projectile motion. A football follows a parabolic trajectory (due to gravity), meaning we can model its motion using quadratic equations to predict its height, distance and overall flight path.



As you know a quadratic curve with a negative coefficient of x^2 does go below the x-$axis$ (as shown in A), however, in the case of the parabolic motion of a kicked football it still follows a quadratic equation with a negative coefficient of x^2, but it does not go below the x-$axis$ (as shown in B) unless you are considering bouncing or an underground reference frame. Therefore, we will be considering a football that starts from the ground, rises, reaches a peak, and then lands back on the ground, staying entirely at $x \geq 0$.

1. If we are told the quadratic equation we can find both the horizontal distance travelled and the maximum vertical height reached by the football.

<u>Example:</u> *It is reported that the football kicked by Shelby Hogan in the 27th minute against Gotham FC followed the quadratic equation $h(x) = -2x^2 + 5x + 7$, where $h(x)$ represents the height of the ball in metres and x representsthe horizontal distance in metres. Find the maximum height reached by the football and the total horizontal distance it travels before hitting the ground.*

<u>To answer the following we are going to:</u>

1. Factorise the quadratic equation
2. Make the quadratic equal to 0 to find the roots.
3. Draw a sketch to help us to visualise the curve.
4. Find the distance between the roots which will give use the horizontal distance travelled by the football
5. Complete the square and find the turning point (the y coordinate is the maximum height) / differentiate the quadratic and make it equal to zero (as at the turning point the first derivative is equal to 0)
6. Give the units (metres) with our answers.

$h(x) = -2x^2 + 5x + 7$

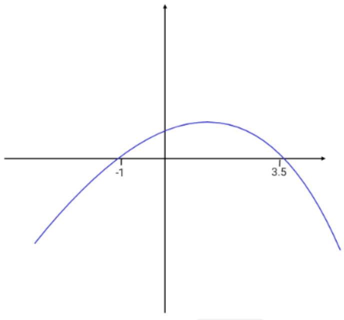
$\rightarrow -2x^2 - 2x + 7x + 7$

$\rightarrow -2x(x + 1) + 7(x + 1)$

$\rightarrow (-2x + 7)(x + 1)$

$\rightarrow (-2x + 7)(x + 1) = 0$

$\rightarrow x = -1$ and $x = 3.5$



$\rightarrow 3.5 + 1 = 4.5$

$\therefore$ Horizontal distance travelled = 4.5 metres

Method 1: Completing the square

$h(x) = -2x^2 + 5x + 7$

$\rightarrow -2\left(x^2 - \frac{5}{2}x\right) + 7$

$\rightarrow -2\left[\left(x - \frac{5}{4}\right)^2 - \frac{25}{16}\right] + 7$

$\rightarrow -2\left(x - \frac{5}{4}\right)^2 + \frac{25}{8} + 7$

$\rightarrow -2\left(x - \frac{5}{4}\right)^2 + \frac{81}{8}$

$\Rightarrow$ Turning point=$\left(\frac{5}{4}, \frac{81}{8}\right)$

$\therefore$ Maximum vertical distance travelled $= \frac{81}{8} = 10.125$ metres

Method 2: Differentiation

$h(x) = -2x^2 + 5x + 7$

$\rightarrow h'(x) = -4x + 5$

$\rightarrow -4x + 5 = 0$

$\rightarrow x = \frac{5}{4}$

$\rightarrow$ Substitute $x = \frac{5}{4}$ into $h(x)$

$\rightarrow -2\left(\frac{5}{4}\right)^2 + 5\left(\frac{5}{4}\right) + 7 = \frac{81}{8}$

$\therefore$ Maximum vertical distance travelled $= \frac{81}{8} = 10.125$ metres

You can also work backwards if you were given the roots of the quadratic or just the turning point as the generic equation of a quadratic is $y = ax^2 + bx + c$ (in the case of modelling the projectile motion of a football the coefficient of x^2 is negative).

Volume of a Football

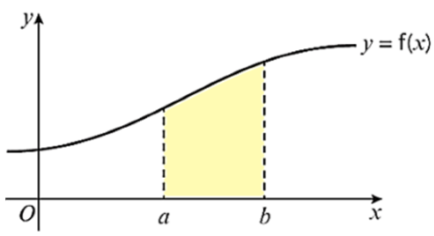
You might remember working out the volume of a sphere in your GCSE Mathematics paper, you may have used the formula sheet or just remembered that it is $V = \frac{4}{3}\pi r^3$.

But have you ever wondered where the equation came from? And how it can be used to work out the volume of a size 5 football? And why a size 5 football provides the optimum volume to enhance performance?

We are now going to use integration (which we looked at in an earlier section) to work out volumes of revolution which in turn will help us work out the volume of a football.

Volumes of Revolution:

We have used integration to find the area of a region R bounded by a curve, the $x\text{-}axis$, and the two vertical lines which is given by the formula: $Area = \int_a^b y \, \mathrm{d}x$ where $y = f(x)$ is the equation of the curve.

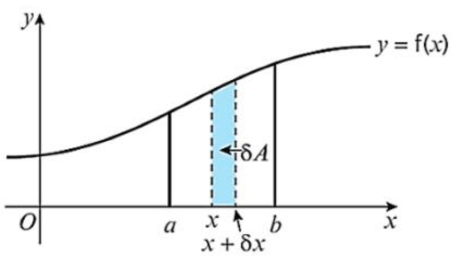


You can derive this formula by considering the sum of an infinite number of small strips with width δx .

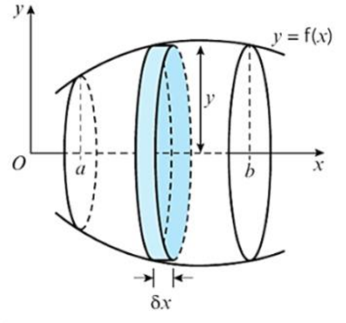
Each of these strips has a height of y, so the area of each of strip is $\delta A = y\delta x$.

The total area is approximated the sum of these strips $\sum y\delta x$.

The exact area is the limit of this sum as $\delta x \to 0$ which is written as $\int_a^b y \, \mathrm{d}x$.



You can use a similar technique to find the volume of an object created around a coordinate axis. If each of these strips is rotated through 2π radians or $360°$ about the $x\text{-}axis$, it will form a shape that is approximated cylindrical. The volume of each cylinder will be $\pi y^2 \delta x$ since it will have radius y and height δx (since the volume of a cylinder is worked out using: $v = \pi r^2 h \rightarrow$ which you learnt during your GCSE Mathematical studies).



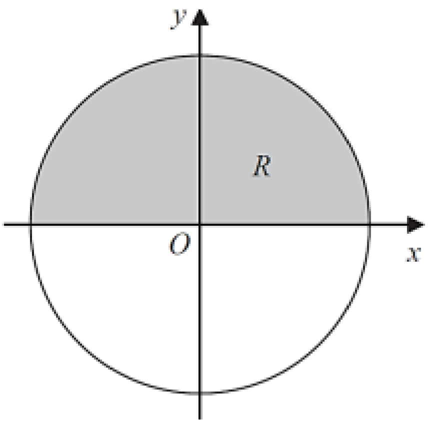
So the volume of the solid will be approximately equal to the sum of the volumes of each cylinder, or $\sum \pi y^2\, \delta x$. The exact volume is the limit of this sum as $\delta x \rightarrow 0$ or $\pi \int y^2\,\mathrm{d}x$

$\Rightarrow$ The volume of revolution formed when $y = f(x)$ is rotated through 2π radians about the $x\text{-}axis$ between $x = a$ and $x = b$ is given by: $V = \pi \int_a^b y^2\,\mathrm{d}x$

Think about a circle with radius r and centre at the origin. What would be its equation?

Let's go through it: remember that the equation of any circle is given by $(x-a)^2 + (y-b)^2 = r^2$ where $(a,\ b)$ is the centre of the circle and r is the radius, in this case as the centre is $(0,0)$ the equation of the circle is $x^2 + y^2 = r^2$.

If I now shade the region R which is the region bounded by the $x\text{-}axis$ and the part of the circle for which $y > 0$, it will look like this:



Now let's use volume of revolution (which uses integration) to rotate region R through 360° about the $x\text{-}axis$ to create a sphere with volume V.

Remember that the equation is: $V = \pi \int_a^b y^2\ \mathrm{d}x$

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

Make y² the subject of the formula

$$\Rightarrow \quad V = \pi \int_{-r}^{r} r^2 - x^2 \quad dx$$

$$\rightarrow V = \pi \left[xr^2 - \frac{1}{3}x^3 \right]_{-r}^{r}$$

Upper limit is r and lower limit is -r

$$\rightarrow V = \pi \left[\left((r)r^2 - \frac{1}{3}(r)^3 \right) - \left((-r)r^2 - \frac{1}{3}(-r)^3 \right) \right]$$

Integrate in respect to x

$$\rightarrow V = \pi \left[\left(r^3 - \frac{1}{3}r^3 \right) - \left(-r^3 + \frac{1}{3}r^3 \right) \right]$$

$$\rightarrow V = \pi \left[\frac{2}{3}r^3 - -\frac{2}{3}r^3 \right]$$

$$\rightarrow V = \pi \left[\frac{4}{3}r^3 \right]$$

$$\rightarrow V = \frac{4}{3}\pi r^3$$

Both women's and men's football use a size 5 football, and the radius is ~ 11cm therefore we can now work out the optimum volume which enables the best aerodynamics, and we can also use this to work out the density of the football using $\rho = \frac{m}{v}$:

$\rightarrow v = \frac{4}{3}\pi(11)^3$b

⇒ The volume of an average size 5 football is 5575.3 cm^3 $(1dp)$

By calculating the volume of a football, we see how mathematical principles that students learn during their GCSEs can shape the very foundation of the beautiful game, as once we have determined the volume we can further explore how this optimal measurement enhances the football's aerodynamics, ensuring the right balance of weight, air resistance and flight stability. The same principle can be used to work out the volume of football boots (as you could use a quadratic to model the top of the boot to then work out which materials to use to maximise performance).

There are still many uses of Mathematics in football that we haven't yet explored, but for now, remember how go out there and play football and remember about gender equality and the Mathematics behind your next kick.

All the diagrams are my own apart from the ones used for the volume of revolution section: the diagram of the circle and the coordinate axis is an image taken of Figure 1 from question 3 from the 2020 Edexcel AS Level Paper 1 and the other diagrams from this section are from A level Further Mathematics Core Pure Textbook .