Zero:
Why it is our most annoyingly useful tool

Jahar Singh Purewal

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Introduction

Zero has been known for roughly **2000 year**s, although this is still a hot topic, the origin of nought is not of my focus. But in fact everything after this, the successes and defeats shared with zero- and inspite of zero- lead us through many fields of maths, from our geometric world to the world of complex numbers.

$$0x = 0, \forall x \in \mathbb{C} \tag{1}$$

This is the single rule powering all polynomials, without which their use would collapse. We use this result to tame the beast that is powers, allowing us to find many easier problems. But how do we PROVE this? (Unfortunately, the analogy of having zero cookies and zero friends will not suffice, no matter how much Siri insists)

1 Proof of the Singularity

Our zero has it's own axiom, within the axioms of the complex, it is declared that 0 is the additive identity $\forall x \in \mathbb{C}$. In normal words, this means that adding 0 doesn't change our equations. Since this applies to app complex numbers, it mar also apply to 0 itself: $0 = 0 + 0$. So we can now equate these two to show (1).

Rewriting 0 as $0 + 0$: (1)

$$0x = (0 + 0)x$$
$$\therefore 0x = 0x + 0x$$

Let $0x = a$, then:

$$a + a = a$$

Using our axioms , we can assert that $a = 0$. Therefore, we conclude $0x = 0$, which holds for all values of x since we assumed nothing about x. The universality of this result allowed us to explore more complex expressions in the early days of mathematics.

2 Implications in Nasty Polynomials

This fundamental result plays a critical role in polynomial equations. For example, consider the quadratic equation:

$$x^2 + 6x + 5 = 0$$

which factors as:

$$(x + 1)(x + 5) = 0$$

Using the zero-product property (1), either:

$$x + 1 = 0$$
$$\textit{or}$$
$$x + 5 = 0$$

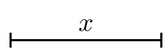
which implies:

$$x = -1, x = -5$$

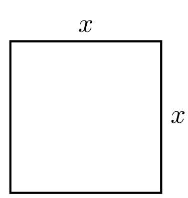
2.1 Geometric Interpretation

The concept of zeros in polynomial functions extends geometrically:

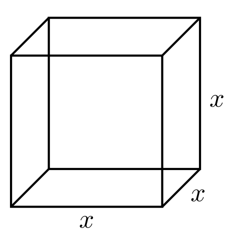
- The linear equation $x = c$ can be visualized as:



- The quadratic equation $x^2 = c$ can be visualized as:



- The cubic equation $x^3 = c$ can be visualized as:



- The quartic equation $x^4 = c$ cannot be visualized as a 4D geometric object.

In each case, we can observe that if the space each object takes, respectively, is 0, then the number of sides that have be zero that cause the object to take up 0 space would be at most n sides, but at a minimum of one must be 0, in a space of dimension n. This pattern is quite reminiscent of the Fundamental Theorem of Algebra, that each polynomial with degree n has n complex roots, and gives us a stepping stone as to how the theorem is true, all from messing around with 0.

For example, the rectangle in 2D has two choices for 0, width or length. (Technically three if we count the trivial case of 0 width and 0 length)

2.2 Dividing by Zero

Now exploring an inverse of our singularity becomes tricky. We can quite easily deduce that 0 has no obvious multiplicative inverse algebraically.
Let x = 0/0, we can then multiply both sides 0:

$$0x = 0, \forall x \in \mathbb{C}$$

Since this is true for all values of x, we can conclude that, in theory, all numbers are the multiplicative inverse of 0. If we were to run with this idea, then logically we can show:

$$\begin{aligned} &1 = \frac{0}{0} \\ &2 = \frac{0}{0} \\ &\therefore \\ &1 = 2? \end{aligned}$$

Obviously this is contradictory, right? Well, it depends, if you want to keep the other axioms (assumed true statements), then yes this is contradictory and so must be thrown out. However, we can just run with it and see where else we end up, just because it contradicts one of our base statements, doesn't necessarily mean we must throw it out. This is a common theme with zero that we have to become comfortable with, working outside the intuitive logic. However we see quickly that dividing by 0 doesn't lead to a useful system where laws and logic can be applied.

If we let p = $\frac{1}{0}$:

$$0p = 1$$
$$2 \cdot 0p = 2$$

Using the associative property of multiplication, we rearrange:

$$(2 \cdot 0)p = 2$$
$$\therefore$$
$$0p = 2$$

Substitue in $0p = 1$:

$$\therefore$$
$$1 = 2?$$

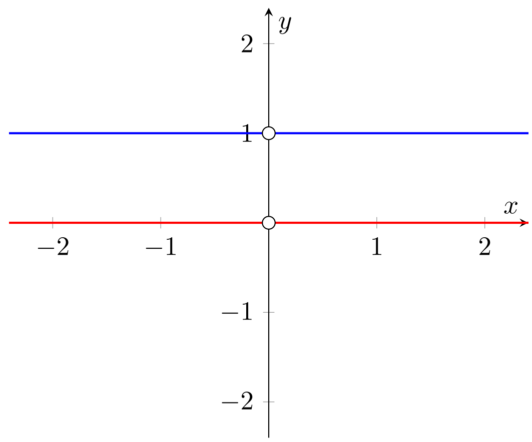
We again arrive at the same conclusion as $\frac{0}{0}$, so it is clear that a system in which we can divide by zero means that all numbers are equal to each other, so technically we can divide by zero, but it's just not very useful.

3 Tetrating the Singularity

Tetration is the repeated use of exponential, denoted with ${}^{x}y$, where ${}^{2}y = y^y$, is not very often spoken about or used in maths, however, it is brought to the spotlight when referring to the singularity.

We first must understand the singularity in exponents. We can easily say that $0^x = 0, \forall x \in \mathbb{C}$ as we have already shown (1) that multiplying by 0 always equals zero, so logically many multiplications of zero must also equal zero.

Likewise, we can show that $x^0 = 1$ as we can extend the series of x^n in reverse, each lower power is equivalent to dividing by x each step, so in the case of $x^1 = x$, if we divide by x, we get $x^0 = 1$.



As shown on the graph above, both of these graphs would never converge. That being due to no intersection at $x = 0$. But also, when we try to use these known facts about 0 and powers, we get contradictory results as the first rule says $0^0 = 0$, but if we first apply then $0^0 = 1$, On the surface, it seems as though 0^0 has no defined value. I will later show a proof for 0^0 equally one of these values, while not a complete proof, is still often used in many fields of maths.

4 Limits, Derivatives, and Folia

4.1 What is a limit?

Well, a limit rigorously encodes the idea of "approaching" something and is denoted with the following symbol:

$$\lim_{x \to h} f(x)$$

Now not so commonly mentioned, but, if we were to take the limit of x as it approaches some value we can evaluate, then it should, and thankfully does, equal the value of the function at that point, in notation that means:

$$\lim_{x \to h} f(x) = f(h)$$

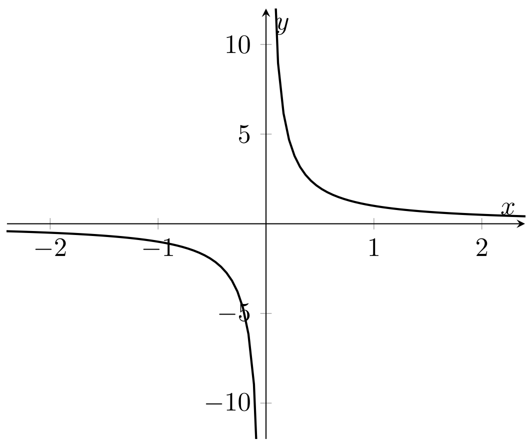
This was our way of double checking that using limits made sense, because on the surface, it doesn't really make sense to do this, why would inputting values that aren't what we are looking for be helpful. Now this is because limits are a fancy way of naturally extending patterns, overcoming limitations of functions. Now this is a tool most often used when zero pops its head around.

As mentioned above, we can use this tool to find sensible values for functions, but not for all, as zero still remains a pesky number.

4.1.1 Cases that don't work

We can try to use the idea of limits to evaluate the multiplicative inverse of 0, but to no avail. Try for example, $\frac{1}{0}$:

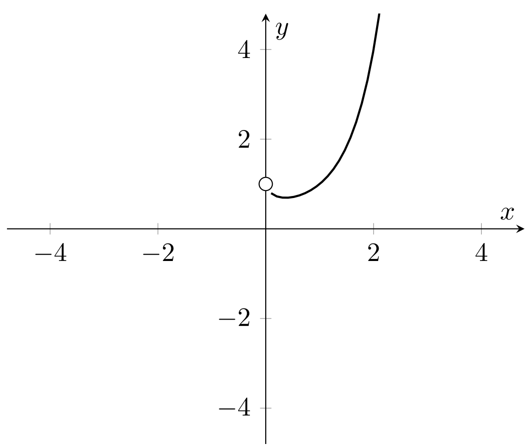
$$\lim_{x \to 0} \frac{1}{x}$$



As is visually obvious, the limit will fail to produce one value as approaching from different direction will give you either $-\infty$ or ∞, this lack of clarity means that the limit does not exist. (Unless a stereographic projection is considered, where $-\infty$ and ∞ are mapped to the same place on the sphere in 3D space, resulting a unsign ∞, however this would be a point of discontinuity).

We can try to find ${}^{2}0$ using the same process, try to find x^x as $x \to 0$.

$$y = x^x$$



As you can see, if we try to evaluate the following limit:

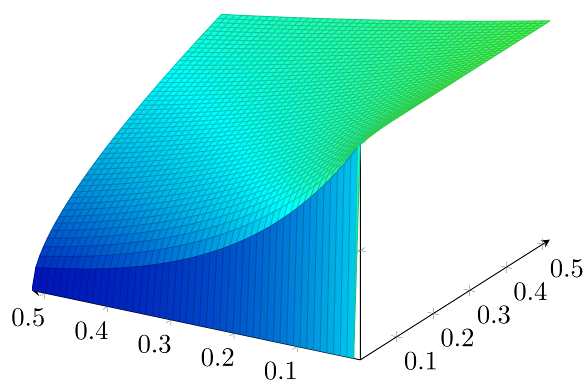
$$\lim_{x \to 0} x^x$$

It appears that the answer would be 1, at least based of the graph above. This could be further supported by another argument, recall earlier how I said that $x^0 = 1$ because we can extend the series x^n by dividing each lower term by x, this is only an approximation of sorts, what is really happening is that each step down we are multiplying by 1 less x, which looks like dividing by x, but isn't necessarily. So we can use the fact that 1 is the multiplicative identity and say that each term of $0^n = 1 \cdot 0^n$, so when we hit 0^0 we can say that we multiple 1 by 0 one less time than we did for $0^1 = 1 \cdot 0$, resulting in $0^0 = 1$.However, the graph does not represent the possibilities for x^x, when computing the limit, we must also consider multi-variable functions.

$$\lim_{(x,y) \to (0,0)} x^y = \text{indeterminate}$$

This limit doesn't exist as when we try to compute this, coming from different directions can result in different results. This makes a limit undefined as we are extending the 2D case, such as we did for $\frac{1}{x}$, a limit cannot be equal to multiple different answers.

$$z = x^y$$

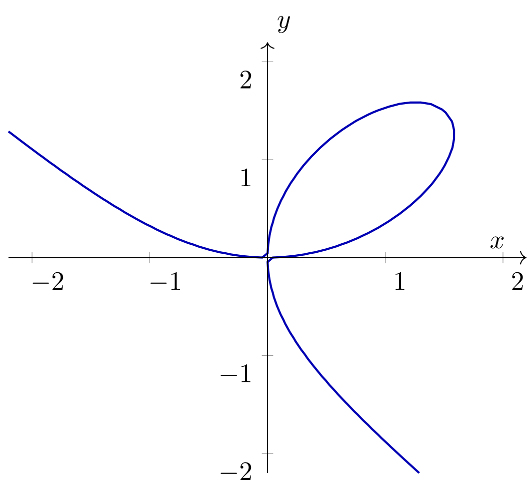


The graph above illustrate my point. The vertical axis (z) is governed by the equation $z = x^y$, so the z-intercept are all the points that satisfy 0^0, now we can take a limit of this function as we previously did by taking a path along this, and since the point we end up at different points based off the direction we take, this limit is not defined . This does come with the caveat that for many cases in algebra, like the Taylor series expansion for e^x and within the binomial expansion require some definition of 0^0 usually being 1.(Since we know that $e^0 = 1$, and the other terms but the first one clearly go to zero, the first term, $\frac{0^0}{0!} = 1$ so $0^0 = 1$)

4.1.2 What about cases that do work?

The most famous case of this working is with derivatives, as a concept they are pretty simple, the represent the rate of change, the gradient, at a point on any continuous curve. The concept of a derivative is simple, but it was first put into practice in a much more informal, yet insightful way, through the FOLIUM OF DESCARTES.

Now what is a folium? The Folia are a family of implicit curves, with the form $x^3 + y^3 - dxy = 0$. Now Descartes, the father of graphical geometry, challenged Fermat (Yes the same guy who said he didn't have enough pages to prove his last theorem) his rival, to find the derivative of a point on this curve-the tangent line problem.



Now Fermat's insight into solving this problem was to notice a pattern shared with the curve and the tangent, the closer the tangent point was to any other point on the curve, the more resemblance these two points had to falling on the same straight as the tangent. So he decided to approximate the tangent initially by working out the gradient of the line between two close points on the folium, then simplifies his expression to ensure we aren't dividing by 0, and then lets the different between the two points become zero.

This is almost line for line what we do with limits and derivatives from first principles. However, Fermat came under heavy judgment because of the lack of rigorous steps that allowed the distances between two points be non-zero while also being zero later on, this is where the Theory of Fluxions (Newton's name for Calculus) took its place.

4.1.3 Derivatives and the Singularity

The derivative is already a powerful concept in its own, combined with 0, can represent unique points on graphs. For example, the derivative being 0 signifies a turning point, a hump or a valley locally. But a more powerful concept is using derivatives and 0 to compute new limits, L'Hôpital's rule.

4.1.4 What is L'Hôpital's rule?

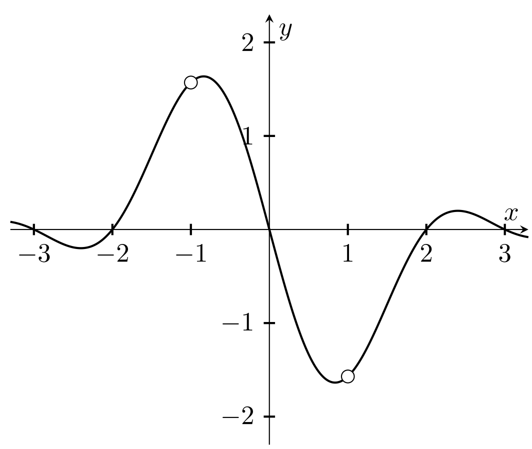
This rule is a way of using limits to compute seemingly impossible limits, and it relies on representing changes in a function as (the gradient) $\cdot$ (small change in x) $= f'(x) \cdot dx$. When we then try to evaluate certain limits, we can use the fact that the dx can be cancelled out.

Graphically, this can be thought of as zooming in at the troublesome point, graphing the denominator and numerator on their own. At this zoomed in perspective, the two graphs begin to look like a straight line, approximately, and so any tiny nudge dx will also approximately be on this line, so at no point are we computing the troublesome function like $\frac{0}{0}$, but can appropriate it with values **very** close to it.

Now we can take a look at an example of this in action and study the consequence of it with the following limit:

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x^2 - 1}$$

Now the graph of this function looks like this:



While the graph is not defined at $x = 1, -1$, it is quite apparent that there is some corresponding result that extends the function to those points continuously. The reason the curve is undefined at both point is because they come out to be $\frac{0}{0}$, which is an indeterminate form. However, L'Hôpital's Rule allows us to use derivatives to evaluate this limit, which is an attempt to evaluate $\frac{0}{0}$. Now L'Hôpital's Rule is a useful tool to reiterate my first problem with 0, it has no unique multiplicative inverse. This is evident as the rule evaluates $\frac{0}{0}$ to be different values in different graphs, this inconsistency helps to reflect why the inverse of 0 isn't useful, it is unreliable.

5 Ultimate Judgment of 0

Despite the simplicity (and emptiness) of 0, it remains a paradoxically core concept of mathematics, as indispensable as addition. As the backbone of products, it holds up the entire field of polynomials through its status as an identity in addition and the singularity nature of it in multiplication. Allowing us to explore concepts in algebra, geometry and calculus. While single-handedly shackling those very same fields in other ways, but only through these shackles could we mere humans counteract these problems, to some extent anyways- breaking our minds in the process. As a enabler and disruptor, zero constantly reminds us of its challenging nature, acting in some ways as a dare to mathematicians, and only through this troublesome number (and question) can these ideas arise.

6 Inspiration

6.1 YouTube:

1. 3Blue1Brown - Channel
2. Another Roof - Channel
3. Numberphile - Channel
4. Dr Sean - Channel
5. BriTheMathGuy - Channel