Normal Distribution: Natural Art

Essay by Smriti Desai for Tom Rocks Math Competition

I can still remember the day I first saw the normal distribution. It was a warm January evening and I was drawing the cover of a farewell card for my Mathematics teacher. I was leafing through my father's copy of George Gamow's *One Two Three... Infinity*, and happened upon an illustration depicting the variation in the probabilities of getting heads or tails, depending on the number of coin tosses. I didn't know it was called the normal distribution, and I didn't know that it was being used to approximate a binomial distribution. I didn't even know about standard deviation back then. Yet, it took me no less than fifteen seconds to understand what Gamow was trying to convey with his graph.

For two coin tosses, there is a 50% chance of an equal number of heads and tails, and a 25% chance each for all heads or all tails. As the number of tosses increases, the probability distribution graph flattens at the edges (because it becomes increasingly impossible for all heads or tails) and spikes up in the middle. It was very simple, and very obvious, and blew fifteen year old me's mind.

There is something really beautiful in the perfect bilateral symmetry of the normal distribution. Humans have been drawn to symmetry for a very very long time (there is an engraving on a wristband from Mesin, circa 11000 BCE with symmetrical designs [{1}]) and I can't claim to be any different. Symmetry has held connotations of order, balance and harmony throughout the centuries, manifesting in art, sculpture, architecture, and music. It is clear in the fresco of *Christ Giving the Keys of the Kingdom to St Peter* (Pietro Peregino) and perhaps less so in the fact that most popular contemporary music uses the symmetrical 4/4 time signature, be it *Smells Like Teen Spirit* by Nirvana or *Shake It Off* by Taylor Swift. We gravitate towards stability. That is why we build homes, why we study mathematics, and why people who find mathematics tricky dislike it.

However, aesthetic allure can only take it so far. It is very easy to wax poetic about this bastion of balance. There is a lot more for interest underneath the veneer of this bell-shaped curve.

The Function

The normal or Gaussian distribution is used to represent a continuous random variable based on only two parameters: the mean (μ) and the standard variation (σ). The basic features of the normal curve are its symmetrical decrease in probability density as values move further away from the mean, and the peak of the curve being its mean, median *and* mode. So, a larger standard deviation leads to a greater spread of the data and vice versa. Very simple.

$$\frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

The above is the function for normal distribution with mean μ and variance σ^2. But to avoid unnecessary complexity, the random variable being studied can be transformed into a standard normal variable, denoted by Z. By substituting $\mu = 0$ and $\sigma^2 = 1$ into the equation for the normal distribution PDF (probability distribution function), we arrive at:

$$\varphi(z) = \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\,.$$

$$Z = (X - \mu)/\sigma$$

The is denoted by the Greek letter $\phi(z)$, where z represents the number of standard deviations from the mean. Any $z < 0$ represents a value less than the mean, and vice versa. This z can be found by dividing the difference between μ and the value by σ. The total area under the graph of $y = \phi(z)$ is 1.

The value of $P(Z \leq z)$ is denoted by $\Phi(z)$. The function's symmetrical nature allows us to calculate $\Phi(z)$ and $\Phi^{-1}(z)$ with only half the values.

And here the normal distribution proves its graceful simplicity. While there are more complex aspects to its working, it is the most intuitive distribution of them all. It feels the most natural, and easily fits in with our understanding of the world, which is why it has applications in a variety of fields from astrophysics to economics.

Applications

Most physical quantities in our expansive world, be it natural or manmade, are continuous. Continuous quantities are prone to random errors—it is impossible to measure continuous values to complete precision. You could be measuring the mass of a carrot or the distance between two stars. Irregardless, you will get a spread of values centering about the 'true' value. In fact, Gauss, to whom the discovery of normal distribution has been attributed [2], used it to judge measurement errors in astrophysics.

Given its possible use for any continuous variable, it is often used as the first approximation[3] , especially in theoretical research. When data is insufficient, the normal distribution can be used to estimate intermediate values due to its omnipresence. Statistical methods can later be used to empirically test this assumption. Common domains include biology (when measuring physical dimensions of organisms), psychology (for Intelligence Quotient and similar tests) and finances (a logarithmic normal distribution is often used for stock prices as they generally remain positive).

Central Limit Theorem

Of course, not everything is a continuous variable. We have already discussed coin tosses, a form of binomial distribution, being represented with a normal distribution curve. The central limit theorem states that under certain conditions (variables must be random , independent, identically distributed with expectation μ and finite variance σ^2), the sum of many random variables will have an approximately normal distribution. This holds true whether the original variables are distributed normally or not, thanks to the law of large numbers allowing the sample average to converge to the expected value μ as n approaches infinity.

In this way, the normal distribution mirrors a vast proportion of the probability distributions used to describe our world. In its symmetry and near-universality, it colours a painting of the various quantities and objects that interact in our world. It is how the natural world reveals itself to us, like how the light bouncing off snow-capped mountains reveals itself to artists, or how the first chirps of songbirds in the springtime reveal themselves to musicians.

References

[1] https://doi.org/10.1142/2847 Symmetry as a Developmental Principle in Nature and Art by Werner Hahn

[2] Probability and Statistics 1 Coursebook by Dean Chalmers, Cambridge University Press

[3] On Lines and Planes of Closest Fit to Systems and Points in Space by Karl Pearson