Particle Accelerators and Electric Fields: the Mathematics of the Unmeasurable

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1 Introduction

The unforeseen proportion of the efforts endured by the physicists responsible for the creation of the very first particle accelerators is generally disregarded. Building machines capable of propelling something as little as electrons - whose sizes are so insignificant that it has been impossible to even accurately measure their diameter[1] - is, by itself, an incredibly impressive idea to perceive, let alone fully develop in times when calculations were exclusively performed by hand and the discovery of nuclear fission was a relatively recent event. Max Steenbeck, Gustav Ising and, above all, Nobel-winning Ernest Lawrence are just a handful of the names behind the creation of such beautiful exemplifications of the utter genius of mankind: these names, forgotten by many, are of those who we are most grateful to for the discovery of every periodic-table element heavier than Uranium, for the progress made in the fields of quantum mechanics and for the advancements that this technology has made possible in almost every aspect of human life, from pure science to medicine and a lot more.

However, this short *excursus* about the *men* will have definitely raised questions about the *thing*: what *is* a particle accelerator? A particle accelerator, as this self-explanatory name suggests, is a contraption that jointly uses electric and magnetic fields - respectively, to accelerate and to bend the trajectory of particles - to *shoot* groups of electrons - which are previously ripped off from their atoms by applying a voltage difference - into linear or circular chambers, where they travel emitting radiation. What is impressive about accelerators is the sheer speed that the electrons inside them are able to reach: we are talking of values that scrape that of light. The number of particle accelerators - between normal ones and *synchrotrons*[2] - is estimated to be approximately 30,000.

2 Structural characteristics

As shown in image *[1]* below, a linear accelerator[3] - if oversimplified - can be thought of as a hollow cylinder: at the end A of the solid figure, an electron gun is placed - which initially accelerates the particles. Around surface Ω, a number of quadrupole and sextupole electromagnets[4] can be found: they are fundamental to *focus* the electron beam, counterbalancing the electric repulsion forces between the particles. Without these magnets, accelerators as we know them would be physically impossible, as the electrons, driven towards the structure of the cylinder, would eventually collide with the walls of the chamber.

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[1] The electron, in the Standard Model, is considered a point particle, thus it has no known size.

[2] Synchrotrons are circular accelerators capable of *synchronizing* the electric and magnetic fields acting on the particles with the particles themselves.

[3] A notable linear accelerator is Italy's *Free Electron Laser Radiation for Multidisciplinary Investigations* (generally shortened to the acronym *FERMI*)

[4] Dipole electromagnets are widely used as well.

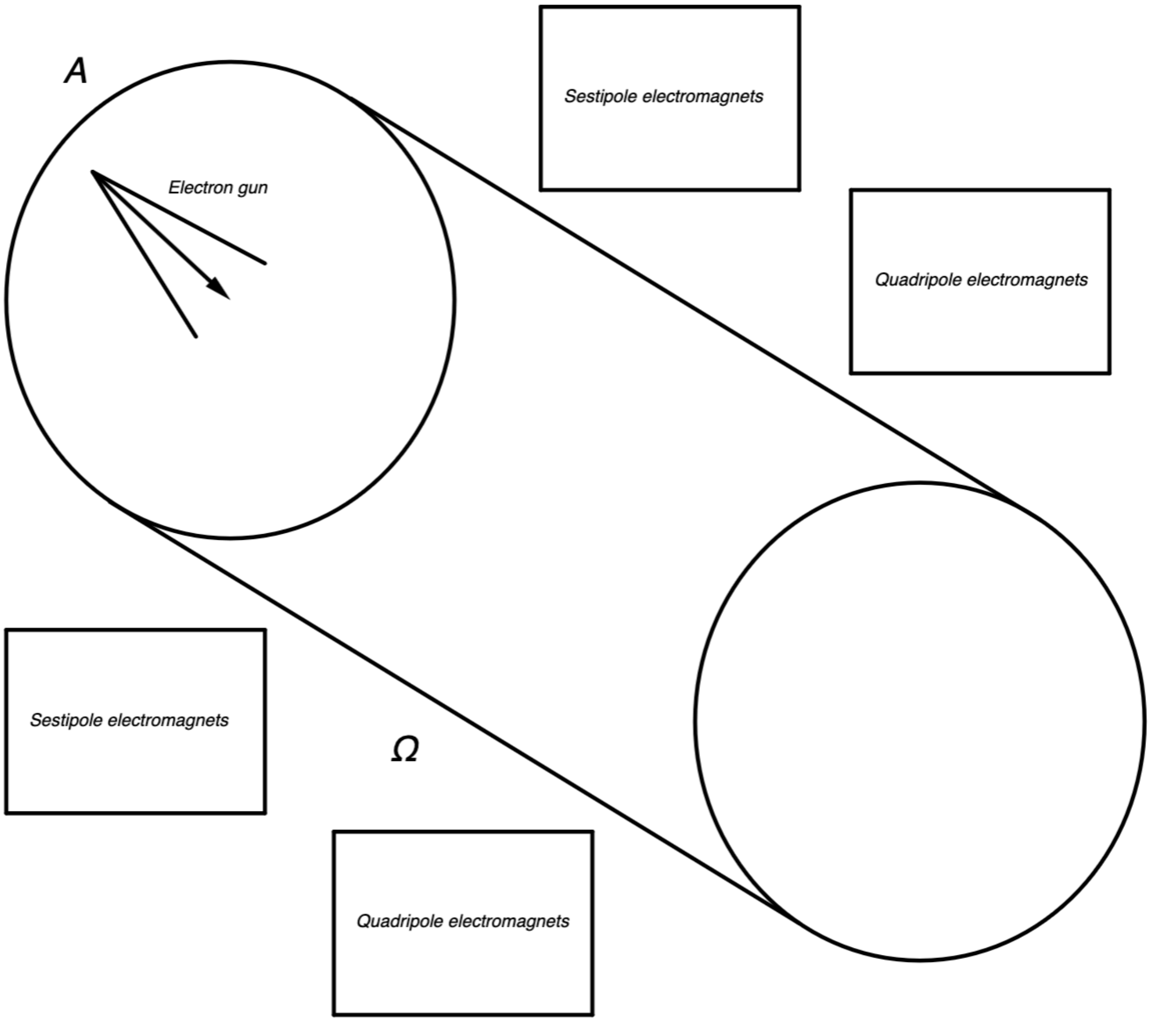


Image [1]

However, the use of magnetic fields to control these trajectories may appear as a counterintuitive solution to a problem that could, otherwise, be solved by electric fields: charged bodies, which share a number of aspects with electrons, seem like the only, most rational way to follow. This raises the question: how could electric fields be used instead of magnetic ones in linear accelerators?

3 The new accelerator

In order to use electric fields as means to prevent the electrons from escaping the center of the accelerator, it is required to make a few modifications to the model previously discussed: the electromagnets should now be removed (or, at least, turned off) and surface Ω is to be considered as a uniformly charged surface of negative charge density σ. The surface needs to be negatively charged in order for it to create forces that repel the electrons, driving them towards the center. Also, in order to simplify the, otherwise, extremely complex calculations, the cylinder should be considered of infinite length.

This assumption is justified by what assuming a cylinder of finite length *x* would mean. As modeled in image *[2]* below, points A, B, C, D, and every other of the points along Ω would impress forces of different magnitude on electron *e*: this would inevitably result in an alteration of the path of the electron, which - to be computed accurately - would require calculations in three dimensions. The only points where this characteristic would be irrelevant are those inside the section of the cylinder at its center (at distance $\frac{1}{2}x$ from either side). Assuming an infinite cylinder would completely erase this problem: if scaled to infinity, the force created by each point would be counterbalanced by that of another symmetrical to the first one.

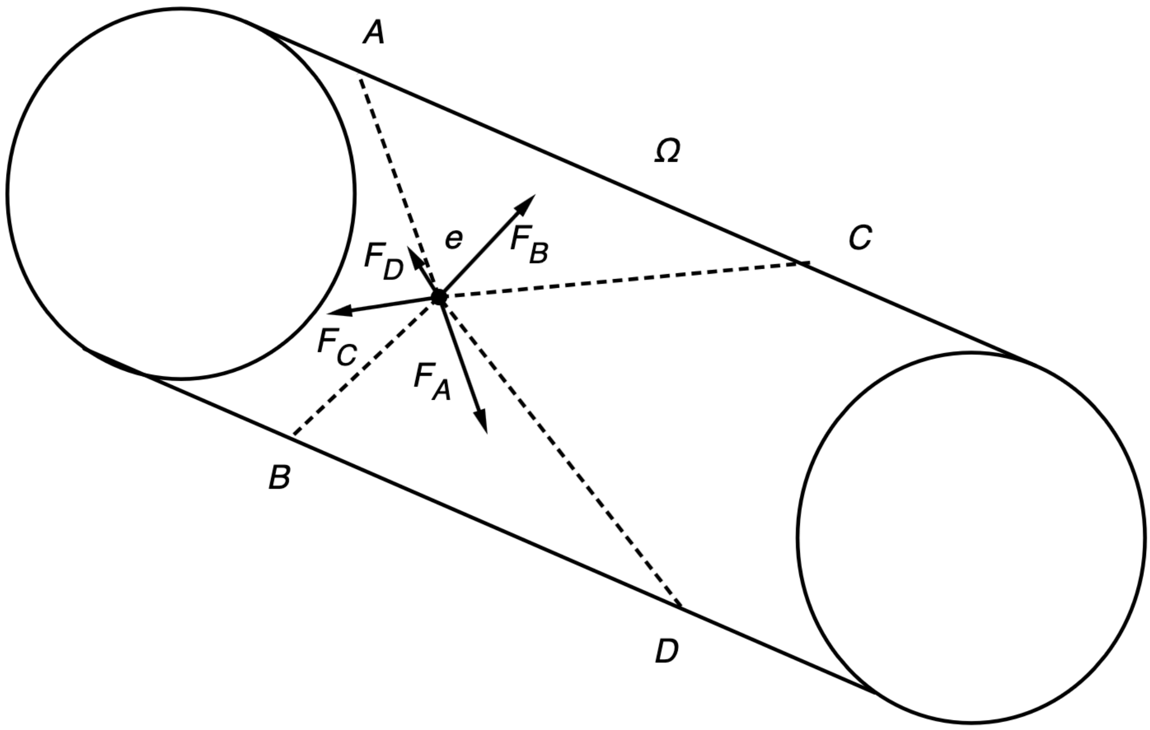


Image [2]

For the record, it is now important to estimate the scale of the effects that this approximation will have on further calculations. To achieve this, it is sufficient to just consider the effects on the motion of electron *e* along the axis parallel to length *x*. This is justified by the fact that only a handful of points on Ω (significantly near electron *e*) impact significantly the path of the charge along the axis perpendicular to *x*.

Inside the chamber, the charges are subject to no air resistance, or other forces impressed by the environment. This enables us to approximate the motion of electron *e* as uniform and linear as it leaves the electron gun: it is now necessary to calculate the module *v* of this initial velocity.

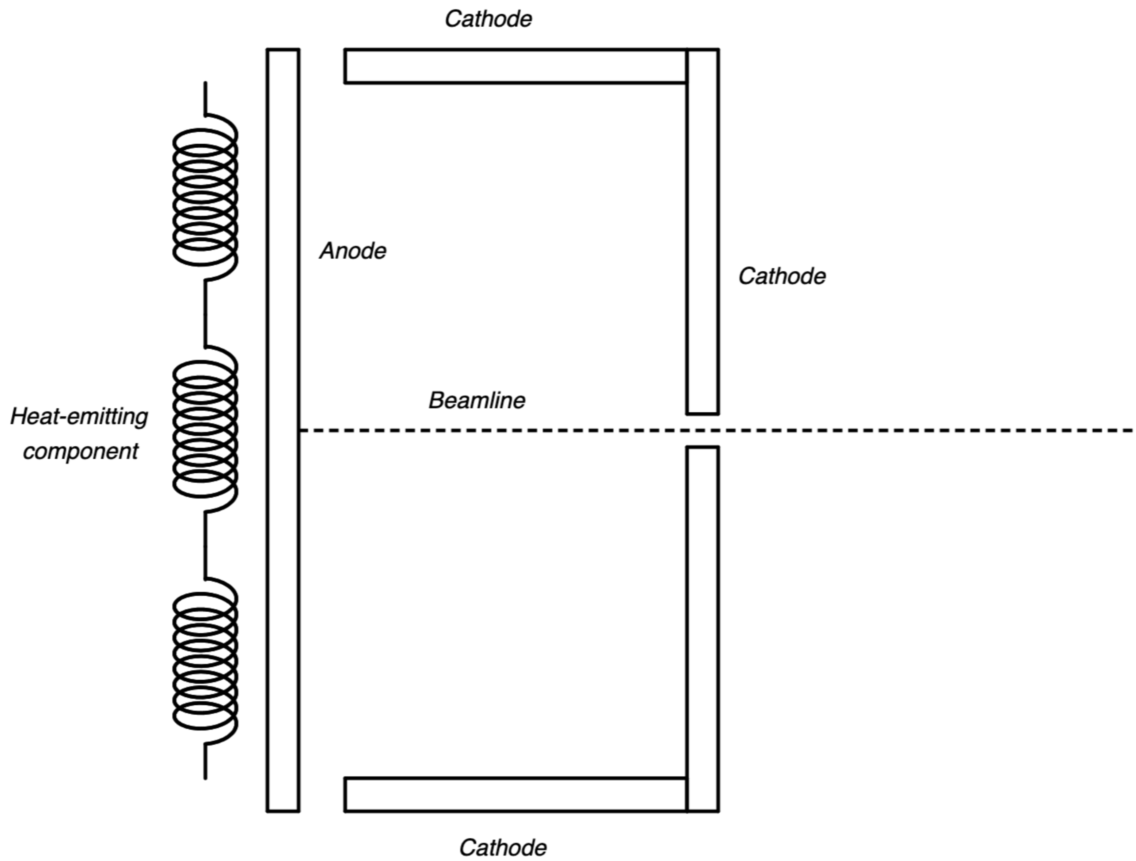


Image [3]

As shown in image *[3]* above, the electron gun is composed of two electrodes: an anode of negative charge and a cathode of positive. Behind the anode is generally placed an heat-emitting component, a key factor in order to accelerate the electrons, which are propelled thanks to a phenomenon called *thermionic effect* and which travel through a beamline crossing the structure of the cathode.

Between the electrodes a potential difference exists: let this be ΔV. The electrons are accelerated thanks to a force of repulsion from the anode and one of attraction to the cathode. The resultant of these forces performs work of value $W_{A \rightarrow C}$. By definition of potential difference:

$$\Delta V = - \frac{W_{A \rightarrow C}}{- C_e}$$

$$W_{A \rightarrow C} = C_e \Delta V$$

[1]

Where C_e is the absolute value of the charge in Coulombs (C) of electron e (1.6022×10^{-19}C).

Now that a way of expressing the work performed by the force in terms of *ΔV* is found, it is fundamental to remember that, by definition of work:

$$W_{A\rightarrow C}=\Delta K$$

$$W_{A\rightarrow C}=K_f-K_i$$

Where *K* is the value of the kinetic energy of electron *e*. Since the electron is, before being accelerated, still, the value of K_i is zero. Therefore, by definition of kinetic energy:

$$W_{A\rightarrow C}=K_f$$

$$W_{A\rightarrow C}=\frac{1}{2}m_e v^2$$

Where m_e is the mass in kilograms (kg) of said electron (9.11×10^{-31}kg). With reference to formula *[1]*:

$$C_e\Delta V=\frac{1}{2}m_e v^2$$

At this point, by rearranging the formula for *v*, an expression for the velocity reached by the electron as it leaves the gun is found:

$$v=\sqrt{\frac{2C_e\Delta V}{m_e}}$$

The only parameter that still needs to be set to a fixed value is that of potential difference *ΔV*, which can vary - depending on the considered accelerator and its purpose - from a few units of kiloVolts (kV) to several megaVolts (MV). In this case, any accelerator could be considered: let the choice fall on the SLC[5], which was generally used at a cathode-anode potential difference of 120kV, but was able to withstand up to 180kV. Let the chosen value of *ΔV* be the pondered mean between the two potentials, with a weight ratio of 2:1 in favour of the normal operational difference: the final obtained value is equal to 140kV.
Performing the required calculations:

$$v=\sqrt{\frac{2\left(1.6022\cdot10^{-19}C\right)\left(140\cdot10^{3}V\right)}{9.11\cdot10^{-31}kg}}\approx2.21910\cdot10^{8}\frac{m}{s}$$

[5] The Stanford Linear Collider (SLC) was part of the SLAC National Accelerator Laboratory and operated from 1989 to 1998.

This value, which is fairly close - as anticipated - to the speed of light in a vacuum, provides an idea of how insignificant the forces applied by the charges distributed along Ω would be in a finite length cylinder, respectively to the motion of each electron on the x-axis. To deeply acknowledge the real magnitude of this statement, it is sufficient to calculate what forces would be needed to slow the electrons down by a factor of 1%. Let the new velocity be *v'*:

$$v' = \frac{99}{100} v \approx 2.166909 \cdot 10^{8} \frac{m}{s}$$

By definition of variation of momentum (*Δp*):

$$\Delta p = v' \cdot m_e - v \cdot m_e = m_e (v' - v) = m_e \Delta v$$

The concept of impulse must now be introduced. By definition of impulse (*Imp*):

$$Imp = \Delta p$$

$$Imp = m_e \Delta v$$

[2]

Impulse is a significantly useful physical idea. Indeed, it is mathematically equal to the area under the time-force curve of a variable function of force *F(t)*, between two considered values of time *t*. Assuming that electron *e* leaves the gun at *t=0s* and that it reaches the end of the linear accelerator at $t = t_f$ (therefore, changing its position by a factor of *x* in an interval t_f), by definition of definite integrals:

$$Imp = \int_{0s}^{t_f} \sum F_x(t)\, dt$$

Where $\sum F_x(t)$ is the resultant of the forces along the x-axis that every point of Ω impresses on electron *e*. With reference to formula *[2]*, the definite integral should be equal to:

$$Imp = m_e \Delta v = m_e (v' - v) \approx -4.754600 \cdot 10^{-24} \frac{kg \cdot m}{s}$$

The reason why the impulse sums up to a negative amount is related to the fact that, in order for the electron to be slowed down, $\sum F_x(t)$ would need to be contrary to *v* for most of time period t_f. At this point, it is safe to consider that the value of the required impulse is still too large for $\sum F_x(t)$ to determine variations as little as a 1% speed change to motion of each electron. As exemplified by image *[2]*, $\sum F_x(t)$ is, indeed, of a slightly significant magnitude, since each point impresses a force in a different direction - most of which, in addition, counterbalance each other -, accounting for a slim resultant force along the x-axis. Moreover, it is clear that the resultant $\Sigma F_x(t)$ will not be always contrary to the motion of the particle: once a certain point along the x-axis (at $\frac{1}{2}x$) is surpassed, the electron, instead of being slowed down, will be accelerated.

Another factor that underlines what already stated is related to time period t_f, during which $\Sigma F_x(t)$ is applied: it is, indeed, too limited to let the momentum change previously described take place. As a matter of fact, taking once again the SLC as an example, the collider was built with a reported length (d) of 3.2km: a rough estimate of t_f, is given by assuming that electron e, during its motion through the chamber, has a mean velocity $\bar{v}$ ($2.19300\times10^8 ms^{-1}$), given by the arithmetic mean between v' and v, therefore:

$$t_f \approx \frac{d}{\bar{v}} \approx 1.45919 \cdot 10^{-5} s$$

This result and the conclusions previously reached show that the approximation of the cylinder to one of infinite length will not impact significantly further calculations. This leads to the next important chapter of our journey to the creation of a new type of accelerators: how the charges along Ω can hold the electrons restricted to the center.

4 Electric fields and accelerators

The infinite-cylinder assumption implies that the only impactful forces on the motion of each electron along the axis perpendicular to length x - which will be, from this moment onwards, referred to as y-axis - are those impressed by the points of Ω that lie on the circumference of the cross-section of the cylinder intersected by the electron.

Before analyzing any specific situation, it is fundamental to derive an expression for the electric field in a given point P_e inside the collider. The calculations will be performed with reference to the model shown in image *[4]* below: on a set of axes with origin *O,* a circumference of radius R and center in point O is plotted. P_e has both of his coordinates set to non-zero values: dealing with two quantities to express the position of the point might result in complex calculations. This problem, however, is elegantly overcome by considering the symmetries of the circle, which make it immune to rotations. This means that, by rotating the whole system in either direction until one of P_e's coordinates is equal to zero will result in no mathematical change. The situation, finally, reached is exemplified by image *[5]*, which shows that P_e has now reached coordinates *(d, 0)*, where d is the distance that, in the first model, existed between the point and origin O.

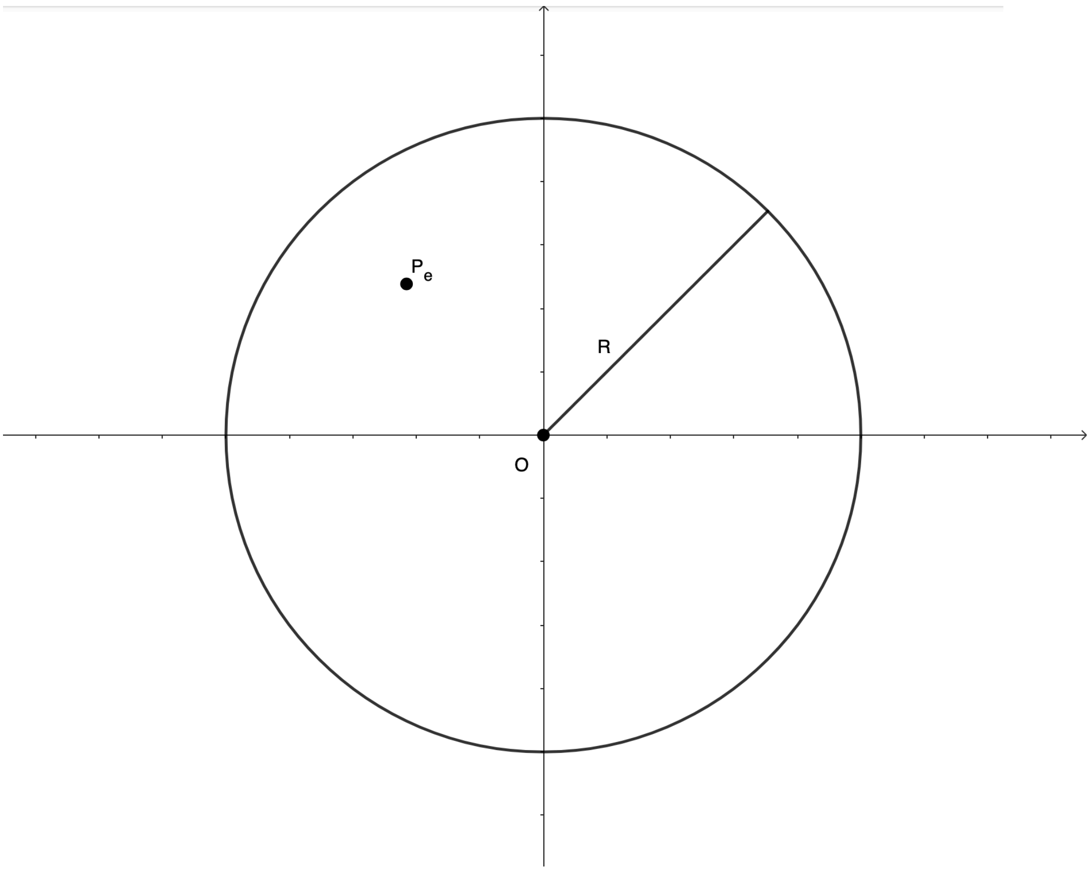
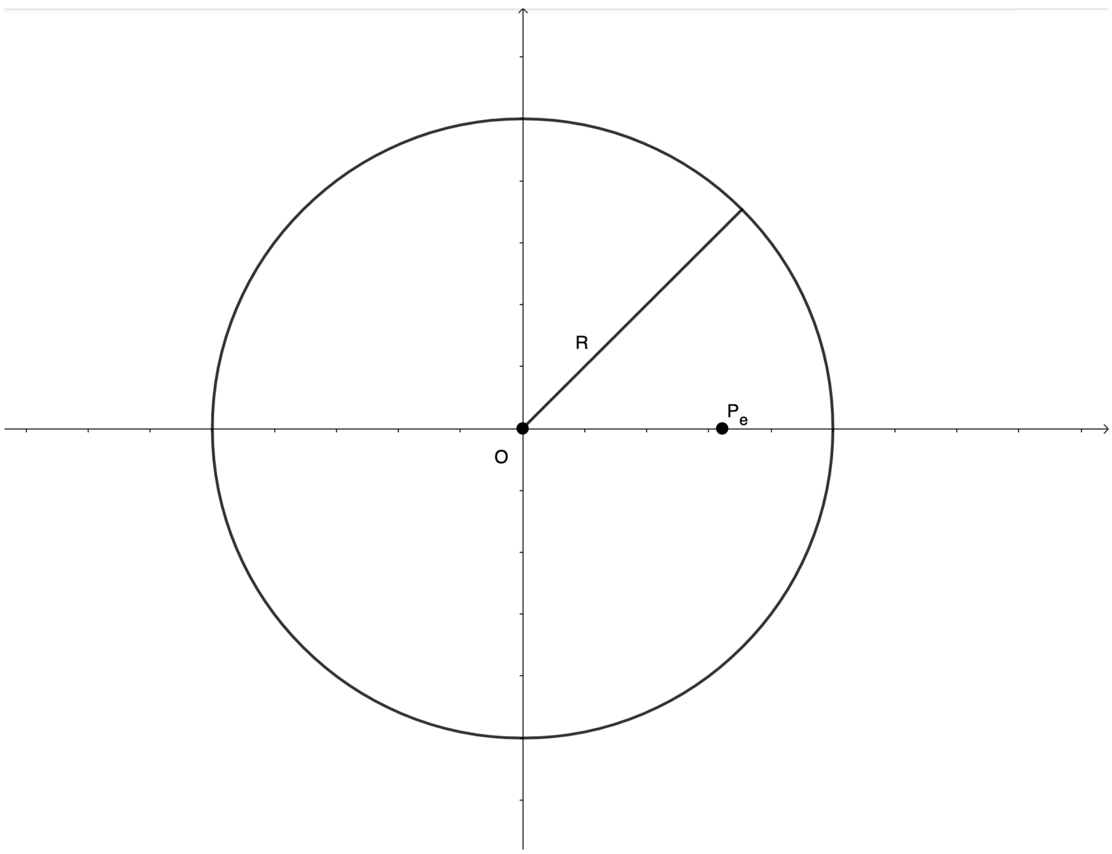


Image [4]



Image[5]

The circumference, as previously underlined, is charged and has a negative linear charge density λ: let the total charge of the figure be Q. The only way to derive a formula, from this moment, is that of calculus and of dealing with infinity: by splitting the circumference in an infinite number of infinitesimal arcs, each of this arc will have length equal to dC and charge equal to dq. By definition of linear charge density:

$$\lambda = \frac{dq}{dC}$$

[3]

Each of the infinitely many dC's will rest on an angle of value (expressed in radians) $d\theta$. By definition of radian:

$$\frac{dC}{d\theta} = R$$

Rearranging the expression for dC:

$$dC = R \cdot d\theta$$

[4]

Image *[6]* provides the very last bit of understanding of the model required to deduct the formula. Point P_e lies on a segment of length r, which connects it to one of the arcs dC - which, being infinitely small, is approximable to a point. Radius R creates an angle of amplitude θ with the horizontal axis. Also, before any deeper mathematics, it is fundamental to acknowledge that no electric resultant force along the vertical axis exists, since the single forces created by the individual points counterbalance each other. This means that the total electric field has only a component along the horizontal axis: let this component be E_x.

Having underlined these key concepts, it is now possible to calculate the electric field dE generated by an arc dC of charge dq. By definition of electric field and for Coulomb's law:

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2}$$

[5]

Where ε_0 is the absolute dielectric constant of the environment inside the chamber, which is a vacuum. Combining formulae *[3]* and *[4]* and reorganizing the obtained expression:

$$dq = \lambda R d\theta$$

With reference to formula *[5]*:

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R}{r^2} d\theta$$

[6]

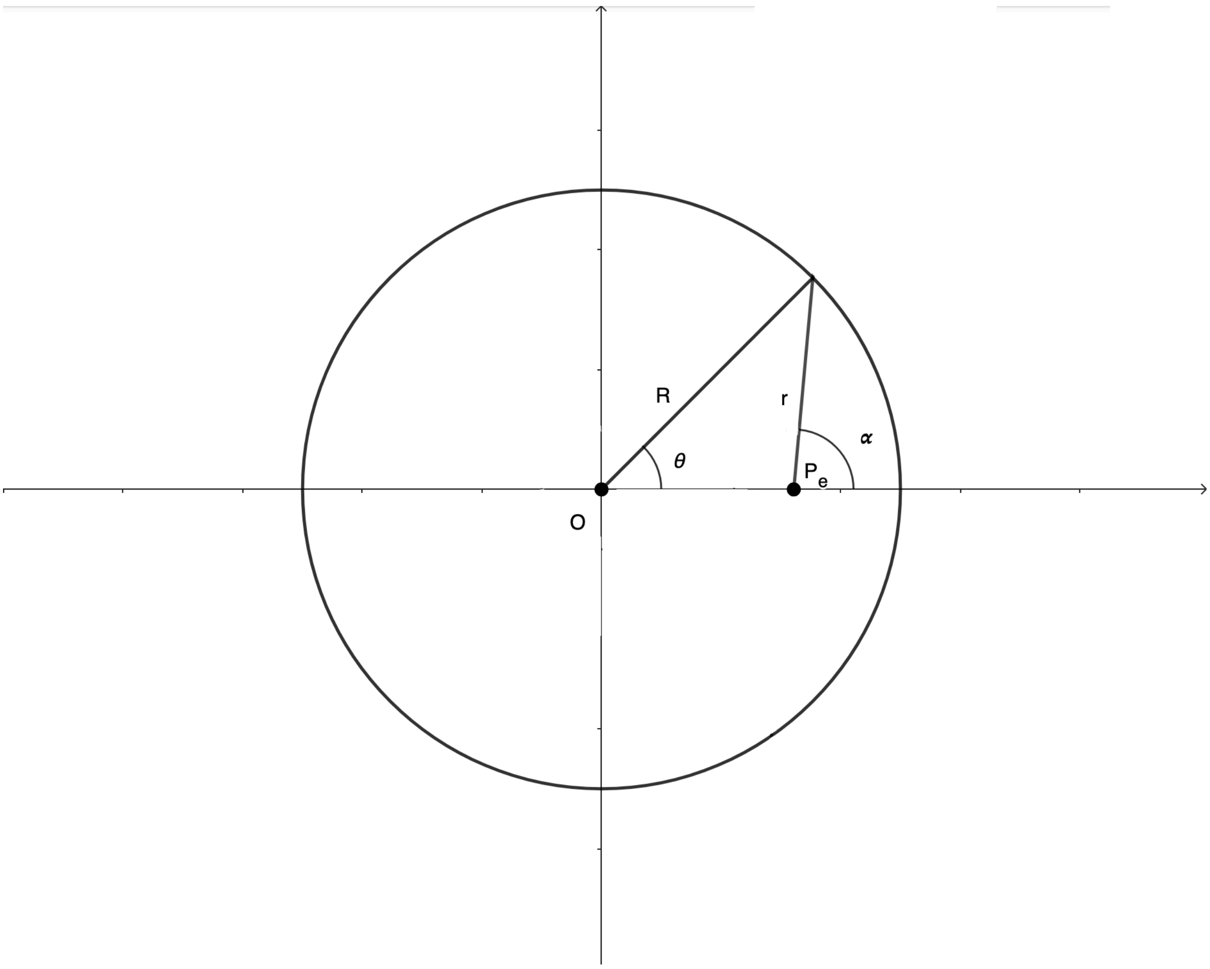


Image [6]

This expression, though useful, still includes a number of variables which would make it utterly impossible to reach a formula to perform accurate calculations. The first variable that needs to be removed from the expression is distance r: this is elegantly achieved by considering the triangle formed by radius R, distance r and the horizontal coordinate d of P_e. For the law of cosines:

$$r = \sqrt{R^2 + d^2 - 2Rd\cos\theta}$$

[7]

This equation now enables us to express r in terms of θ. From formula *[6]*, it is obvious that an integration will be required in order to derive a definitive expression. However, as anticipated, when integrating, the resultant vertical component will amount to 0: this grants the possibility of integrating dE_x instead of dE. For the properties of right-angled triangles:

$$dE_x = dE \cdot \cos\alpha$$

[8]

However, this choice now makes it necessary to achieve a way of expressing the cosine of α - which is the angle between *r* and the horizontal axis - in terms of *θ*. By applying, once again, the law of cosines to the triangle previously considered for the derivation of formula *[7]*:

$$R^2 = r^2 + d^2 - 2dr\cos(\pi - \alpha)$$

$$R^2 = r^2 + d^2 + 2dr\cos(\alpha)$$

Rearranging for *cos(*α*)*:

$$\cos(\alpha) = \frac{R^2 - d^2 - r^2}{2dr}$$

With reference to formula *[7]*:

$$\cos(\alpha) = \frac{2Rd\cos\theta - 2d^2}{2dr} = \frac{R\cos\theta - d}{r}$$

[9]

Everything should now be ready for the anticipated integration. Thanks to formulae *[6], [8],* and *[9]*:

$$dE_x = \left(\frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{r^2}\right)\frac{R\cos\theta - d}{r}d\theta = \frac{\lambda R}{4\pi\varepsilon_0}\frac{R\cos\theta - d}{r^3}d\theta$$

For the fundamental theorem of calculus and formula *[7]*:

$$E_x = \int_0^{2\pi}\frac{\lambda R}{4\pi\varepsilon_0}\frac{R\cos\theta - d}{r^3}d\theta$$

$$E_x = \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{R\cos\theta - d}{\left(\sqrt{R^2 + d^2 - 2Rd\cos\theta}\right)^3}d\theta$$

[10]

The limits of integration are *0* and *2π*, since the intention is that of computing the electric field generated by all the charges distributed along the circumference: a full turn occurs when going from *0* to *2π*.

The form of the obtained integral is obviously complex: articulate calculations would be required in order to reach an antiderivative and insidious integration techniques - such as elliptical integration - would be necessary. For this reason, it is preferable to find an alternative way to solve this integral, which, though resulting in an approximation of some sort, will be able to return fairly accurate results, making, at the same time, unnecessary any complex mathematics.

Analyzing the integrating function, the numerator would require straightforward integration techniques, while the denominator is a fairly complex expression to work around. This shows that the most rational way to simplify the function is that of modifying the form of its denominator. The first step in this direction is removing the square root: at first glance, the expression under the root resembles the expansion of a binomial raised to the second power. However, this is not the case due to the presence of *cos(θ)*: a different approach must be found. To do that it is fundamental to group the expression for R^2 as follows:

$$R^2\left(1+\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta\right)$$

Taking the square root of the reached expression reveals an interesting possibility:

$$\sqrt{R^2\left(1+\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta\right)}=R\sqrt{\left(1+\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta\right)}$$

Indeed, binomial approximation can be used in this case. This type of mathematical procedure is an extremely useful technique (derived from Taylor series), that enables to approximate the square root of any binomial in the form of $\sqrt{1+z}$, with $|z| < 1$, as follows:

$$\sqrt{1+z}\approx 1+\frac{z}{2}$$

In the considered case, z would be equal to:

$$z=\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta$$

Since, for this procedure to work, $|z|$ must be less than 1 and should approach zero as much as possible, this approximation is a perfect fit for the case of linear particle accelerators, as the electrons not only travel through chambers of scarcely significant diameter, but they also move at distances notably close to the center, accounting for very small values of d (with $d \ll R$) and, therefore, for values of z tending to 0. For these reasons:

$$R\sqrt{1+\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta}\approx R\left[1+\frac{1}{2}\left(\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta\right)\right]\approx R\left(1+\frac{d^2}{2R^2}-\frac{d}{R}\cos\theta\right)$$

For the assumption previously made, $\frac{d^2}{2R^2}$ is a slightly significant value: thus, it can be removed, simplifying once again the expression. The final approximation for the square root is:

$$R\left(1-\frac{d}{R}\cos\theta\right)$$

Looking back at the integral inside formula *[10]*, the obtained approximation should now be raised to the power of -3. This power elevation can be performed thanks to, once again, the binomial approximation: in the case of binomials raised to a power γ, in the form of $(1+z)^{\gamma}$ with $|z| < 1$ and $|\gamma z| << 1$, their value is approximable as follows:

$$(1+z)^{\gamma} \approx 1+\gamma z$$

Applying this useful technique to the considered case:

$$\left[R\left(1-\frac{d}{R}\cos\theta\right)\right]^{-3} \approx \frac{1}{R^3}\left(1+3\frac{d}{R}\cos\theta\right)$$

Now that everything is correctly sorted out, it is sufficient to remove the denominator of the integrating function in formula *[10]* and insert the final approximation:

$$E_x \approx \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{R\cos\theta-d}{R^3}\left(1+3\frac{d}{R}\cos\theta\right)d\theta$$

This integral, though seemingly frightening, should be actually solvable by every student that has a grasp on the basics of calculus. After a handful of integral manipulation, the following form should be obtained:

$$\frac{1}{R^2}\int_0^{2\pi}\cos\theta d\theta-\frac{d}{R^3}\int_0^{2\pi}d\theta+\frac{3d}{R^3}\int_0^{2\pi}\cos^2\theta d\theta-\frac{3d^2}{R^4}\int_0^{2\pi}\cos\theta d\theta$$

From this point onwards only straightforward procedures are required. Indeed, by picturing the graph of *cosθ*, when integrating with respect to *θ* from *0* to *2π* the first and fourth integrals evaluate to 0. For this reason:

$$E_x \approx \frac{\lambda R}{4\pi\varepsilon_0}\left[-\frac{d}{R^3}\int_0^{2\pi}d\theta+\frac{3d}{R^3}\int_0^{2\pi}\cos^2\theta d\theta\right]$$

The first of the integrals in the equation is easily solvable via the power rule. For the second one, instead, a substitution must be applied. In particular:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Performing the necessary and elementary final calculations, the following final formula will be reached:

$$E = E_x \approx \frac{\lambda R}{4\pi\varepsilon_0}\frac{d\pi}{R^3} = \frac{\lambda d}{4\varepsilon_0 R^2}$$

[11]

Deriving this elegant and simple expression was a long but, nevertheless, fruitful journey: indeed, given any point P_e near the origin *O* and with horizontal and vertical coordinates $P_{e,h}$ and $P_{e,v}$, it is possible to have an estimate of the electric field caused by the charged circumference in that point.

Although the formula only gives information about the module of *E*, for the symmetries of the circumference, it is obvious that the direction of the field will be towards the center and that the vector will lie on the straight line connecting origin *O* and P_e. In addition to that, it makes perfect sense that at distance *d=0*, the electric field will be, indeed, equal to zero, as all the vectors generated by each *dC* will completely counterbalance each other.

5 From theory to reality

Now that a way of calculating the electric field is found, the answer to the question asked at the very beginning is being approached. What is now required to do is find a way of obtaining a specific value of *λ* for an accelerator to work. Once again, the calculations will be performed based on the structure of the SLC, whose diameter varied from 15mm to 25mm. Let the chosen value of *R* be half of the arithmetic mean of the two diameters: the final radius of the considered chamber is 10mm (10^{-3}m).

To compute the value of *λ*, some other assumptions must be made. First of all, the group of electrons is now to be thought of spherical shape: the center $O_{\mathcal{L}}$ of the imaginary sphere - which will be, from this moment onwards, referred to as $\mathcal{L}$ - is placed along the straight line, inside the cylinder, crossing the center *O* of each section. Let the radius of the sphere be *r'*. In second place, inside the sphere 50 billion electrons (5×10^{10}) should be placed.

For the accelerator to properly work and, therefore, for the electrons to remain inside the boundaries of the imaginary sphere, the electric field ($E_{\mathcal{L}}$) on the surface of the solid figure generated by the electron cloud must be, at least, counterbalanced or, if possible, overpowered by the field (*E*) generated by the charged circumference.

The assumptions previously made enable to easily find an expression for $E_{\mathcal{L}}$. To do that it is necessary to introduce the concepts of electric flux and of Gauss's theorem for the electric field. The electric flux is the scalar product between an electric field vector and a surface vector:

$$\phi(\vec{E}) = \vec{E} \cdot \vec{S} = E \cdot S \cdot \cos\beta$$

In the expression above β is the angle that exists between the two vectors and its cosine is essential in order to compute the value of the flux.

Mathematician and physicist Carl F. Gauss, following the works of J.L. Lagrange, formulated in 1835 a different way of calculating the electric flux of an arbitrary closed surface:

$$\phi\left(\overrightarrow{E_{\mathscr{L}}}\right) = \frac{Q_{tot}}{\varepsilon_0}$$

Q_{tot} is equal to the total charge cointed inside the surface and ε_0 is the dielectric absolute constant of the considered environment. These two formulae are now extremely useful to compute $E_{\mathcal{L}}$. Indeed, equating and adjusting them to work in the considered scenario:

$$E_{\mathscr{L}} \cdot S_{\mathscr{L}} \cdot \cos\beta = \frac{Q_{tot}}{\varepsilon_0}$$

Since, as shown by image *[7]*, no angle exists between any vector $E_{\mathcal{L}}$ and the correspondent vector $S_{\mathcal{L}}$ (where $S_{\mathcal{L}}$ is the surface of $\mathcal{L}$ and, therefore, equal to $4\pi r'^2$), angle β has an amplitude of 0 radians, which means that its cosine is equal to 1 and can be factored out of the equation. Q_{tot} is the total charge inside $\mathcal{L}$ and is, as anticipated, $-50\times10^9\times C_e$. Rewriting the equation for $E_{\mathcal{L}}$:

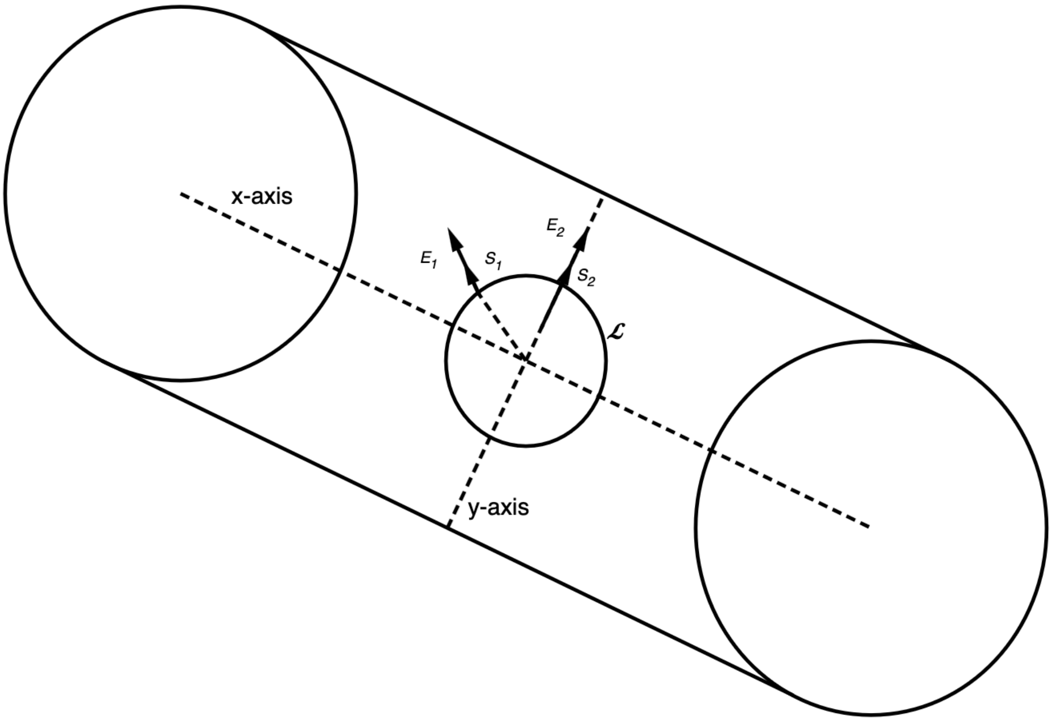
$$E_{\mathscr{L}} = \frac{Q_{tot}}{4\pi r'^2 \varepsilon_0}$$

[12]

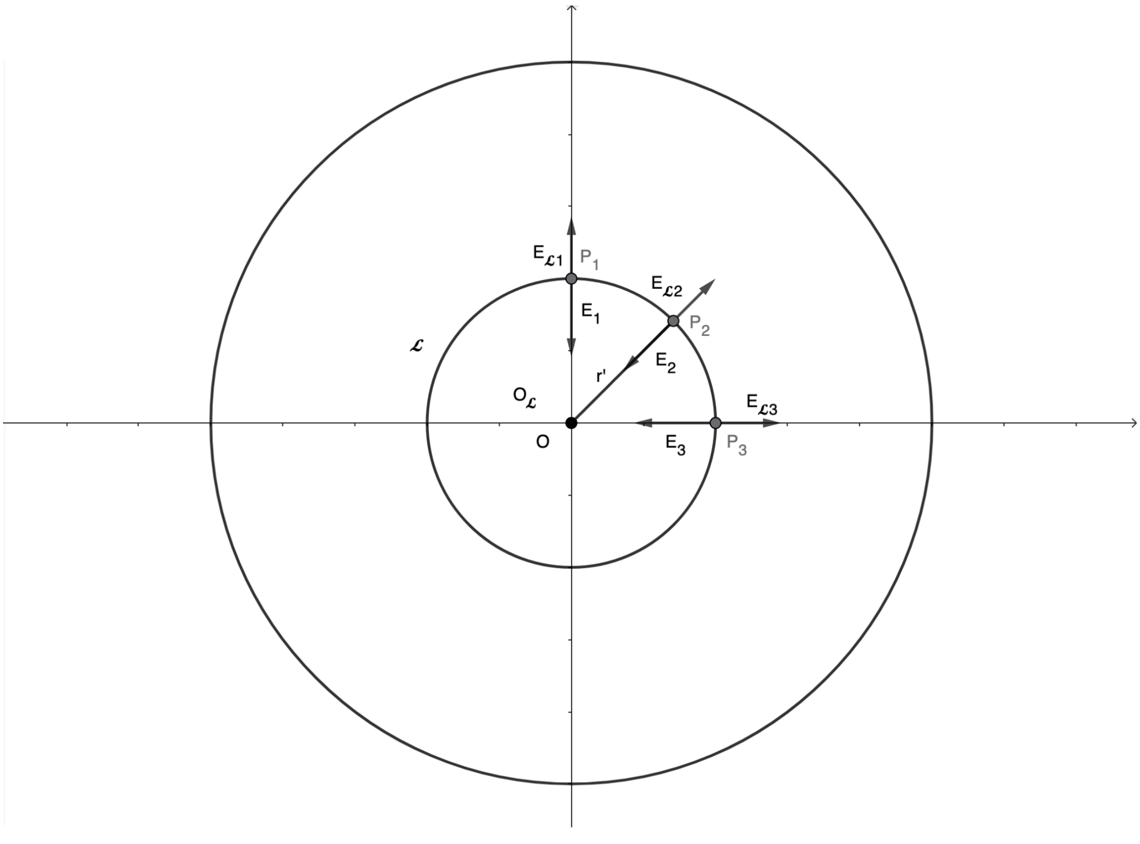
Images *[7]* and *[8]* clearly underline that the points along $\mathcal{L}$ where $E_{\mathcal{L}}$ reaches the maximum value of its component along the y-axis, are those contained by the vertical section of the sphere of radius equal to *r'*. Also, in these points, vector $E_{\mathcal{L}}$ lies on the same straight line - but with opposite direction - as vector *E* (generated by the charged circumference). For this reason, in order for the accelerator to work and for the electrons to be contained inside the arbitrary sphere, the following inequality must be satisfied:

$$\frac{|\lambda| r'}{4\varepsilon_0 R^2} \geq \frac{|Q_{tot}|}{4\pi r'^2 \varepsilon_0}$$

[13]



Image[7]



Image[8]

The reason why, in the previous formula, the absolute value operator is used, is related to the fact that field E needs only to be greater in magnitude: thus, the signs of λ and of Q_{tot} (which are both negative) are not only irrelevant, but would also break the inequality. Rearranging for $|\lambda|$:

$$|\lambda| \geq \frac{|Q_{tot}|R^2}{\pi r'^3}$$

[14]

The radius *r'* should now be set to the radius of the beam travelling across the chosen accelerator: after corrections, for the SLC this value was approximately $2\mu m$ (2×10^{-6}m). Performing the required calculations:

$$|\lambda| \geq \frac{|-50\cdot 10^9\cdot 1.6022\cdot 10^{-19}C|(10^{-3}m)^2}{\pi(2\cdot 10^{-6}m)^3} \approx 3.1875\cdot 10^2\frac{C}{m}$$

As known, λ must be of negative sign, therefore the first of the values that would make the accelerator work is:

$$\lambda \approx -3.1875\cdot 10^2\frac{C}{m}$$

The end of the journey is now extremely near: what there is left to do is draw some conclusions and understand what this number tells us. However, before diving into that, it would be a great idea to put in practice the calculations performed.

6 Building the accelerator

Despite being an extremely fascinating possibility, building a particle accelerator from scratch would not really be the easiest of things. For this reason, to test the mathematics, the only rational option is that of letting a machine compute long and complex calculations and return a rough simulation of how the electrons would behave.

This is nothing that cannot be done by a simple 300-lines Python script, which was uploaded to this GitHub repository: linac accelerator. The code, which is far from complex, has some flaws, which depend on a series of factors. First of all, in order to be able to clearly show the position of each electron an unrealistically small accelerator should be built. In particular the one considered has a radius of 1.5×10^{-12}m, which is comparable to the length that a thousand protons would amount to if stuck next to each other. In second place, the way the model works is by constructing a grid composed of individual 10 femtometers (10×10^{-14}m) wide cells. For this reason, movements - in any direction - smaller than this threshold are not considered by the system: a significant part of the total interaction experienced by each electron is, hence, lost. Also, due to the grid structure, one cell can only be occupied by a single electron at a time: though extremely unlikely, it is necessary for this simulation to work. Lastly, between every check run by the computer a time of 8 picoseconds in the simulated environment passes: this is the shortest interval that enables the machine to produce simulations. As a result, the model does not precisely describe the physical interaction between the involved elements, but only reaches a rough estimate.

Due to these characteristics, it is obvious that quantities of electrons as big as 50 billion could never even possibly be computed by the script. For our simulation, let 4 electrons be inside a cloud of radius ten times smaller than that of the accelerator. Calculating the required values of λ for the system to work through formula *[14]*:

$$|\lambda| \geq \sim \frac{\left|4C_e\right|\left(1.5 \cdot 10^{-12} m\right)^2}{\pi\left(1.5 \cdot 10^{-13} m\right)^3} \approx 1.3500 \cdot 10^4 \frac{C}{m}$$

The script requires inserting the total charge Q of the ring:

$$|Q| \geq \sim -2\pi R|\lambda| \approx 1.2723450 \cdot 10^{-7} C$$

Let the chosen value of Q be -1.3×10^{-7}C. By loading the value and running the script, the simulated environment appears on the terminal window and the electrons start moving inside of it: it is possible to appreciate these results by watching the video uploaded in said repository or by running the script, after having checked that the prerequisites contained in the *requirements.txt* file are satisfied.

One easily notable aspect is that the motion of the particles is more chaotic than expected and they all tend to come closer to the surface than they should. This is independent from the calculations and is a consequence of the structure of the simulation which, as anticipated, is too simple to provide any particularly accurate result. However, the likelihood of our model is just sufficient to acknowledge that - even for relatively long running times - the electrons remain inside the chamber and only some of them ever hit the structure of the accelerator: this means that the work previously done is fairly precise.

7 Conclusion

Both the simulation's results and the mathematical modelling of the hypothetical situation clearly indicate that particle accelerators entirely built through electric fields are possible. However, they would be extremely complicated to put in practice: for the case of the SLC, a considerable charge would have been needed to maintain the linac operational. In particular for a single ring:

$$|Q| \geq \sim 2\pi R|\lambda| \approx 2,010C$$

To put this into perspective, a normal lightning bolt carries an average charge of 15C (with maxima of 350C in the most extreme scenarios): assuming that the total charge amount was to be transferred from a bolt to the structure of the accelerator, just seven of the countless rings would have a charge similar to that unleashed by a strike. In addition, articulate engineering would be fundamental in order to build systems capable of holding charge density σ constant: slight variations of this value might, indeed, compromise the whole structure, which means that even the exposure to quantities of external charged particles and, in general, the environment on our planet would be fatal.

Countless other observations and studies could be undertaken about the potential exclusive use of electric fields in accelerators and the likely outcomes of this choice: this short theoretical analysis just explored the realm of conceptually possible, however not that of practically doable. Although the calculations and simulation show that the described model could work, still a number of unanswered questions remain: this is where the utter beauty of mathematics, physics and science lies. As heard in Christopher Nolan's *Oppenheimer* (2023), *"theory will only take you so far"*.

References

All of the shown images (created through GeoGebra) and the code of the simulator are my own.

The information about the SLC were obtained via the following papers:

- *"Beam-Beam Deflections as an Interaction Point Diagnostic for the SLC"*, Dr. P. Bambade and Dr. R. Erickson, 1986.
- *"Long Ion Chamber Systems for the SLC"*, Dr. J. Rolfe, Dr. R. Gearhart, and Dr. R. Jacobsen, 1989.
- *"High-current, short-pulse, RF-synchronized electron gun for the Stanford Linear Accelerator"*, Dr. C.K. Sinclair and Dr. R.H. Miller, 1981.
- "Accelerator Physics Measurements AT THE Damping Ring", Dr. L. Rivkin, Dr. J.P. Delahaye, and Dr. K. Wille, 1985.

Deeper information about the used mathematical techniques can be found at the following pages:

- Binomial approximation: https://en.wikipedia.org/wiki/Binomial_approximation
- Taylor series: https://en.wikipedia.org/wiki/Taylor_series
- Gauss's flux theorem: https://en.wikipedia.org/wiki/Gauss%27s_law

Other sources:

- Elettra Sincrotrone Trieste: https://www.elettra.eu/

- SLAC National Accelerator Laboratory: https://www6.slac.stanford.edu/
- International Atomic Energy Agency: https://www.iaea.org/
- *Particle accelerators*, Dr. Robert R. Wilson and Dr. Raphael M. Littauer, Zanichelli, 1964.

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