Redefinition of the Super Logarithm Function Invention of the "UTW" Function

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Summary

The inverse of is often a tetration called a super root or super logarithm. Tetration is defined as the hyper-4 operation following exponentiation. It is mathematically described as n iterations in base a. The super logarithm is the inverted tetration. It allows us to know how many iterations were done while reaching the tetration value of the number. In my project, as a new representation of tetration numbers, I have found a new super logarithm representation "UTW" function that makes it easier to operate on super operation numbers, which I have associated with the PTG function (PTG function was presented by me at the International Conference Mathematical Modeling with Modern Applications M3A_24 held by Yıldız Technical University in Istanbul in 2024). I have also developed a new software program "UTWCalc" that can calculate and inverse titration numbers. The reason for writing this program is that the tetration numbers, which are defined as googology numbers, are too large to be calculated on a standard computer and there is no program that I can show these numbers by reducing them. With the UTW function I invented, I can easily calculate the inverse of all super operation numbers (the super logarithm function used today can only calculate the tetration numbers). With the UTWCalc program I wrote, the super logarithm of the tetration numbers, which are among the super operation numbers, is reduced by taking the super logarithm.

The number ${}^{3}8$ ($8^{16777216}$) was calculated with the programme I wrote. The size of the number I calculated is approximately 6.01452075365139E+15151335. In the project, the mathematical induction method was used in the UTW function and the Waterfall method, which is frequently used in software projects, was used in the UTWCalc programme. Another aim of the function I found and the programme I developed is to increase interest in pure mathematics and googology. The research subject is the field of development of mathematical theories and the function I found and the programme I developed increase the functionality of mathematics and facilitate the use of large numbers in natural sciences. In this respect, the project differs from similar mathematics projects.

Keywords: UTW, UTWCalc, Super logarithm, Super operation numbers, Googology

Contents

1. Introduction

Throughout history, mathematics has been a magic wand given to man to make sense of nature. Just like in fairy tales where the mystery of every object touched by fairies is solved, mathematics analyses nature with the axioms set by the human mind and guides man in understanding nature. For this reason, the development of mathematics and natural sciences has turned into a paradoxical structure surrounding each other. As mathematics developed, the limitations of natural sciences were removed, as natural sciences developed, new axioms were accepted in mathematics and mathematical calculations developed in proportion.

Today, technology shows a faster development compared to the past periods. Starting in the 18th century with the Industrial Revolution, the mathematical knowledge that created Industry 1.0, which enabled the use of water and steam energy in factories, has reached the mathematical knowledge that created Industry 4.0, which enables the connection between cyber-physical production systems and physical and digital systems in the 21st century. The partnership of mathematics and natural sciences, which enables the development of these new technologies, has enabled rapid progress in the fields of Internet of Things, artificial intelligence, digital technologies, automation, big data and space exploration. (Kaptangil, 2023, p.5) The developments in recent years have increased the importance of studies with large number sets in mathematics.

Large sets of numbers are the subject of pure mathematics and the branch of science that deals with these numbers is called googology and the scientists working on this field are called googologists. Even Google, a search engine used in many countries of the world today, is named after this branch of science [1]. Although the field of googology is a new field, it has actually existed since humans have been interested in large numbers. It is thought that the first studies on this subject belong to Archimedes in the 3rd century BC. Archimedes was interested in the multiplicity of certain objects based on the difficulty of counting, he developed a new way of calculation.$10^{8x10^{16}}$(Brandshaw, 2001, pp.7-10). This number can be calculated until then was the largest number. In the 19th and 20th centuries, with the emergence of modern mathematics and the invention of the computer, mathematicians were able to access larger numbers than ever before.With the 21st century, working with very large numbers in big data and space research the development of new functions that facilitate the representation and use of these numbers and the development of computer programmes that enable faster solution of large number operations on standard computers or supercomputers.

[1] https://about.google Date of access: 12.09.2024

When the domestic and international literature of the last decade is reviewed, it is seen that the studies in which various calculations based on logarithmic growth, which is used only as a process in different disciplines, are generally in the fields of science and engineering. At the end of the literature review, it was found that the studies were especially matched with software fields such as stellar growth, gravitational theories and cyber security. However, no function that enables the use of the super logarithm function, which is defined as the exponential function of the logarithm, in operations related to other super operation numbers, or software programme that can perform operations by reversing the operations performed with tetration numbers with super logarithmic calculations has been found.

2. Aim of the Project

The 21st century has been the century of science. In this century, many inventions, from artificial intelligence and machine learning to space tourism, have emerged as a result of scientific and technological development. The increasing rate of scientific development is closely related to how the axioms of mathematics are used and how the hierarchy of operations is applied to nature. It is already predicted that scientific development in the next century will be faster than in the past century. Therefore, scientific development in the future will need more new fields of mathematics and new discoveries in mathematics. Large number sets is an area of pure mathematics that is widely used in areas such as big data, nano-technology, quantum physics, bio-technology, computer, astronomy and space sciences. For this reason, in our project, which we have developed based on the needs of the 21st century, a new function that defines the super logarithm and other exponential operations has been found, and a new programme that can calculate the tetration numbers and take the super logarithm of these numbers has been written. *With the functions available today, only the super logarithm of tetration numbers can be taken from super operation numbers, while with the invented UTW (Untowerer) function, the inverse of other super operation numbers such as pentation, hexation, heptation, octation, ennation, decation can be easily taken. In this respect, UTW function is a new invention in the field of pure mathematics. In addition, with the UTWCalc (Untowerer Calc) programme we have developed, tetration numbers can be calculated and the numbers can be reduced by taking the super logarithm.*

3. Conceptual Framework

In this chapter, the tetration operation and the PTG function, which is one of the basic concepts of Googology used to express gigantic numbers, will be explained, and the inverse operation of tetration, the super logarithm, will be discussed in this context. In addition, these operations are performed on large

By comparing the processing power of supercomputers used for large-scale calculations with the processing power of standard computers, the reasons why standard computers cannot calculate large numbers will be explained and the readers will have a better understanding of the functionality of the UTWCalc programme.

3.1. Working Principles of the Tetration Process and PTG Function

3.1.1. Tetration Process and Working Principles

Tetration *is a function* derived from the words tetra (four) in Latin and iteration (repetition) in English, called hyper-4 in mathematics and *valid only for positive integers*. It is a super-operation number, first discovered by the British mathematician and philosopher of science Reuben Louis Goodstein (1912-1985), which allows us to calculate large numbers[2]. Goodstein worked in the field of Finitisim, which accepts the existence of finite mathematical objects, and defined operations such as pentation (hyper operation5), hexation (hyper operation6), heptation (hyper operation7), octation (hyper operation8), ennation (hyper operation9), decation (hyper operation10), which are a series of hyper operations besides tetration (hyper operation4)[3].

The reason for using the concept of tetra in the origin of the word is that the fourth step after the exponentiation operation, which is the third step of the over operator, corresponds to the tetration operation [4]. The steps are defined as addition, multiplication, exponentiation and tetration. It is defined as a repeated exponential operation (Paulsen, 2019, p.243).

: is denoted by a+n. To add both numbers, the symbol is loaded on the numbers that were originally repeated.

Multiplication $a \times n = \underbrace{a+a+a \ldots\ldots\ldots +a}_{n}$ is denoted by "a" itself and "n" times product (repeated addition).

Exponentiation: $a^n = \underbrace{a \times a \times a \ldots\ldots\ldots \times a}_{n}$ "a" multiplied by itself "n" times multiplication).

Tetration Operation: The tetration operation is represented as ${}^{b}a$ in the Goodstein function. (Goodstein, 1964: 17) "For any pair of non-negative integers (a,b), the tetration

[2]For Goodstein's definition and function of superoperation, see also Goodstein, R., L. (1964). Studies In Logic and The Foundations of Mathematics: Recursive Number Theory, Ed: L. E. J. Brouwer, E.W. Beth, A. Heyting. North Holland Publishing Company, Amsterdam.
[3]Hartley, R. (1958). Review: R.L. Goodstein, Mathematical Logic, Bulletin of the American Mathematical Society 64(1):32-35.
[4]Erarslan, M.K. (2023). A New Function for Tetration: "PTG", TUBITAK 2204b Research Projects 2023- 2024.

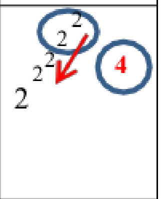
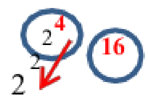
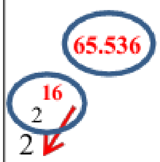
We call this operation a 4th order hyper operator and it is recursively defined follows;" (Ripa, 2024, p.1)

$${}^{b}a = \begin{cases} 1 & \text{If } b = 0 \\ a^{({}^{b-1}a)} & \text{If } b \geq 1 \end{cases}$$

As can be seen, the fourth operation in the hierarchy of operations after addition, multiplication and exponentiation is tetration. When tetration is calculated as *a power with exponent*, the exponent is first taken at the deepest level.

In order to better understand the working principle of the tetration operation, exponential and tetration operations are explained by comparing them on Table 1.

Table1: Exponentiation and Tetration Comparison

Exponentiation		**Tetration Process**	
Operation	5^2	Operation	${}^{5}2$
It is the repeated product of 5 twice.	5x5 = **25**	The upper form of the operation is represented by taking 4(5-1) exponents of 2.	$2^{2^{2^{2^{2}}}}$
Operation	2^5	The process is completed by starting from the topmost exponent and proceeding in the direction of the arrow by multiplying the number 2 by itself. 2x2=4 Now the exponent the number is 4.	
2 is the repeated product of 5 times	= **32**	2x2x2x2x2 = 16 Now the exponent of the number is 16	
As can be seen, the result of 5^2 is not the same as the result of 2^5. Therefore, order is important when dealing with exponents. An exponent is repeated multiplication. Multiplication is similar to repeated addition. However, it is not the same.	$5^2 \neq 2^5$	2x2x2x2x2x2x2x2x2x2x2x2x2x2x2x2 = We get the number 65,536.	
		When we continue multiplying, we reach a tetration number. The resulting number is a super operation number. It is very difficult to represent and requires a super computer processor to calculate it.	 **Number of Tetrations**

3.1.2. PTG Function and Working Principles

Since the UTW function is the inverse operation of the PTG function, it is necessary to get acquainted with the properties of the PTG function. The PTG function[5] is a new function that allows faster and easier analysis of operations with tetration numbers.

[5] In this section, the PTG function will be explained, and the UTW function I invented for the super logarithm operation, which is a continuation of the invention, will be explained in the next section.

function[6]. Apart from the PTG function, the most commonly used functions and their representations in calculations with tetration numbers are given in Table 2.

Table2: Functions Most Commonly Used in Tetration

Function Name	Function	Description
General impression	${}^{n}a$	Introduced by Mourer and Goodstein.
Knuth "s up arrow representation	$a \uparrow\uparrow n$	Number of arrows indicates the amount of expansion
Conway chained Ok demonstration	a→n→2	Conway Right the sequence of positive integers indicates with the correct arrow
Ackerrman Function	${}^{n}2= A(4, n-3)+ 3$	It is the spelling of a = 2 according to the Ackermann function.

PTG function is a more useful and functional function than the functions shown in table 2[7]. In addition, PTG function is used to calculate hyperoperation numbers such as pentation, hexation, heptation, octation, ennation and decation. The notation and rules of PTG function are as follows. (The rules of PTG function are explained with Knuth "s up arrow demonstration).

The rules of Power Tower Generator are as follows;

(I) $ptg[x,y]= {}^{y}x \ (x\uparrow\uparrow\uparrow y)$

(II) The tower quantity of ptg[x,y] =y-1.

(III) $\underset{z}{ptg}[x,y]= x\uparrow^{z}y$

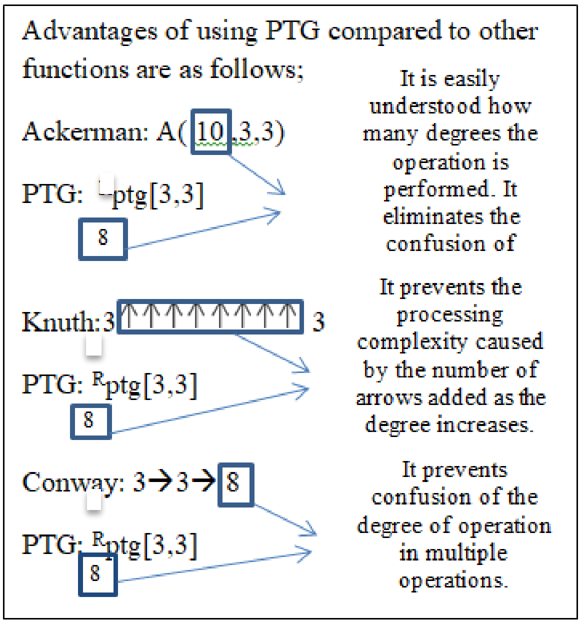
(IV) If Z isn't present, tetration is done.

(V)[8] $\underset{1}{ptg}[x,y] = x^{y}$, $\underset{0}{ptg}[x,y]= x*y$,

$\underset{-1}{ptg}[x,y]= x+y$, $\underset{-z}{ptg}[x,y] = utw_{z-1}(x,y)$

3 tetrad 3 according to rules I and IV;

$ptg[3,3] = 3\uparrow\uparrow\uparrow 3$ (I and IV)



[6] Mehmet K. Erarslan (2024, 4-6, Jun) Power Tower Generator: Invention Of A New Notation For Tetration. [Konferans Sunum Özeti]. M3A_24 International Conference on Mathematical Modeling with Modern, Applications, Istanbul, Türkiye.

[7]The functioning of the functions in Table 2 is beyond the scope of this project and will not be discussed here. For the properties of the functions given in Table 2 and the differences of the PTG function, see Erarslan, M.K. (2024). A New Function for Tetration: "PTG". TUBITAK 2204B Research Projects. Ps.8-16. / Mehmet K. Erarslan (2024, 4-6, Jun) Power Tower Generator: Invention Of A New Notation For Tetration. [Conference Presentation Abstract]. Pp.23. M3A_24 International Conference on Mathematical Modelling with Modern, Applications, Istanbul, Turkey.

[8]Utw is a function of the super logarithm. In the next section we will explain how this function works.

= 3↑(3↑3)
= 3↑27
= 3^{27}
= $7{,}6256X10^{12}$.

3 Pentation 3 according to Rule III:

$_{3}$ptg[3,3] = 3↑↑↑↑↑↑3   (III)

In the pentation formula $_{3}$ptg[3,3], the notation makes the operation more understandable. (3) → The number gives the degree of operation of the operation -2.

= 3↑↑ 7.625.597.484.987

=Cannot be calculated. (The UTWCalc programme written was created to calculate tetrad numbers. It does not calculate pentation numbers. Pentation operation is hyper-5" operation).

As can be seen from the examples, the PTG function does not only enable tetration operation. In the same way, the pentation operation and the operations with other hyperoperation numbers can be done easily. Table3 below shows the use of the PTG function in the calculation of other hyperoperation numbers.

Table3: Use of PTG function in the calculation of other hyperoperation numbers

Operation Rating	**Calculation Method**
Hexation (hyperoperation6)	$_{4}$ptg[x,y]
Heptation (hyperoperation7)	$_{5}$ptg[x,y]
Octation (hyperoperation8)	$_{6}$ptg[x,y]
Ennation (hyperoperation9)	$_{7}$ptg[x,y]
Decation(hyperoperation10)	$_{8}$ptg[x,y]

3.1.3. Super Logarithm and Working Principles

In mathematics, logarithm *is a mathematical function* which is the inverse of exponential functions and *used only for positive integers* (Hobson, 1614, pp.7). Founded in the 17th century by John Napier, the logarithm function soon used by many scientists who wanted to calculate quickly with large numbers. Napier "s invention is as great as the Indian invention of the founder of our number system (Hobson, 1614, pp.5). The logarithm function, which has been used for a long time with calculators, can now be calculated with scientific calculators. Logarithm is generally defined as "the use of a base to obtain a given number.

is defined as the exponent or power to be raised.[9]". In this case, x, $y^x = n$ is the logarithm of n in base y and is written as $\log_y n$. For example, $3^2 = 9$; therefore, 2 is the logarithm of 9 in base 3. In general notation, $2 = \log_3 9$.

Logarithm is the inverse function of exponential numbers, which operates on exponential numbers. It minimises the exponent and makes it easier to perform operations with the exponent. However, it cannot be used in the case of larger numbers, that is, in the numbers obtained as a result of taking the exponent of exponential operations. Because this operation is a tetration operation and is called hyperoperation-4 operation. Therefore, a different operation was needed to minimise the tetration numbers and the super logarithm function was developed. Super logarithm is called iterated logarithm. It is denoted as $Slog_y(x)$ and is read as *super logarithm x in base y*. There are different studies on the super logarithm function [10]. However, the most common usage and rules are as follows.

$$Slog_y(x) = \begin{cases} 0 & \text{If} \leq 1; \\ 1 + Slog_y(\log_y x) & \text{If} > 1 \end{cases}$$

If you want to take the super logarithm of a number calculated with the PTG function, the order of operations should be as follows;

According to Rules I and IV, when the 8 tetration is the super logarithm of 3,

$$8 \text{ tetration } 3 \sim= 6.014521 \times 10^{15151335}$$

$Slog_8(8 \uparrow\uparrow 3) = 1 + Log_8(6.014521 \times 10^{15151335} = 16777216$ In its simplest terms, Slog tells us how many times to logarithm until we reach 1. According to this

$$Slog_8(8 \uparrow\uparrow 3) = 1 + Slog_8(Log_{(8)}(6.014521 \times 10^{15151335})) = 1 + Slog_8(16777216) \quad \text{(Step I)}$$

$$= Slog_8(Log_8(16777216) + 1 + 1 + 1 = 2 + Slog_8(8) \quad \text{(Step II)}$$

$$= Slog_8(Log_8(8)) + 1 = 0 + 3 = 3 \quad \text{(Step III)}$$

The number logarithmised 3 times until it reaches 1. The result is 3.

The super logarithm of the number of tetrations defined as hyper-4 is taken as shown above. For the calculation of the approximate number of tetrations and for taking the super logarithm of this number we used UTWCalc software

[9] https://www.britannica.com/science/logarithm Access Date: 12.01.2025

[10] See Alon, N. and Azar, Y. (1989). Finding an approximate maximum. *SIAM Journal on Computing*, *18*(2), 258-267./ Leiserson, C. E., Rivest, R. L., Cormen, T. H. and Stein, C. (1994). *Introduction to algorithms* (Vol. 3). Cambridge, MA, USA: MIT press./ Devillers, O. (1992). Randomisation yields simple o(n log⋆ n) algorithms for difficult ω(n) problems. *International Journal of Computational Geometry & Applications*, *2*(01), 97-111.

has been developed. The UTWCalc programme can easily calculate the tetration numbers normally calculated by supercomputers on a standard computer. The difference from the previously developed PTGCalc programme is that it can reduce the numbers by taking their super logarithm. Before moving on to the features of the UTWCalc programme, a review of the features of standard and supercomputers will make the project easier to understand

3.1.4. Comparison of Standard Computers and Supercomputers

20. The computer, which is one of the most important inventions of the century, is one of the most important inventions of the century, which is used not only in scientific studies but also in all areas of our lives. Computer is defined as "A device that can perform arithmetic and logic operations according to the programme loaded into its memory, make decisions according to the result and change the flow in the programme, interact with its environment , and keep the input and result data in its memory; electronic brain[11]" in the Turkish Language Association dictionary. Computer has developed rapidly in the last century.

Looking at the history of computer development, it is seen that the first mechanical calculator with moving parts was the abacus developed in China in 2400 BC (Kara, 2013:8). However, by the 17th century, the ease of calculation provided by the abacus was no longer sufficient for mathematical operations. Thus, Pascal developed the Pascal Calculator in 1642. In 1672, the German mathematician and philosopher Leibniz developed the Leibniz Penknife, which calculates faster than the machine developed by Pascal and can perform division, multiplication, square root operations as well as addition and subtraction [12].The first calculator close to the present day was developed by Herman Hollerith. He invented the punched card key for information collection and storage. His invention was used in the history of calculation for about 100 years [13]. The first programmable computer was the Z3 by Konrad Zeus, who lived between 1920 and 1995. Konrad Zeus is therefore known as the father of modern computers (Bock and Eibisch, 2010:713). The Z3 was used for encryption purposes during World War II. The device built in 1941 used the binary number system as well as the decimal number system and was the first computer to use 0 and 1. Developed in 1946 during World War II by the Americans John Mauchly and J. Presper Eckert, ENIAC is a computer that calculates where the bomb dropped during the war will fall. This computer could move 5,000 times a second.

[11] https://sozluk.gov.tr/ Access date: 16.01.20025.
[12] https://bilimgenc.tubitak.gov.tr/makale/bilgisayar-kim-ne-zaman-icat-etti Date of access: 16.01.2025
[13] See. https://www.columbia.edu/cu/computinghistory/hollerith.html Access date: 16.01.2025.

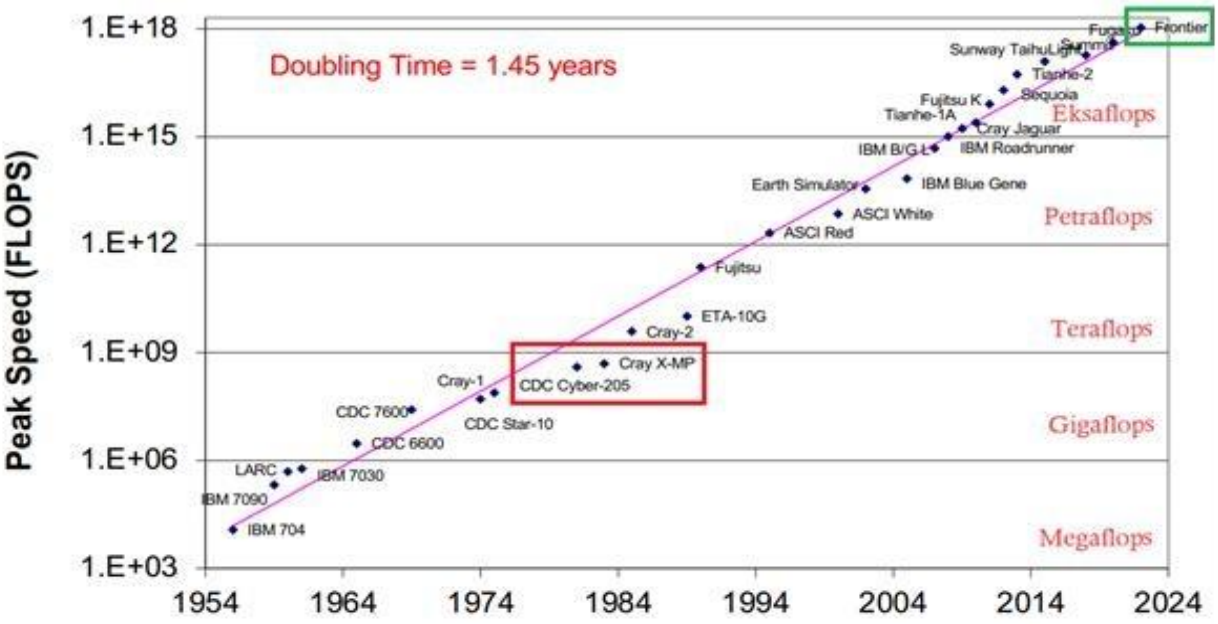
(Weik, 1961: 574) . ENIAC has many hardware features that we use in today "s computers. After the 1950s, with the shrinking of computers and the emergence of the first personal computers, a rapid progress was seen in computer science. In the 1960s, the first supercomputer was the Univac Larc, built for the US Navy and atomic research. The processing power of Univac Larc was 250 kFLOPS. The most powerful supercomputer in the world today is Frontier, built by Hewlett Packard Enterprise (HPE), with a processing power of 1,194 exaflops. There are about 500 known supercomputers in the world.

Today, computers with high processing power compared to classical computers are called supercomputers. Supercomputers consist of a large number of processor cores, large memory capacity and high speed data storage units [14]. These features enable them to perform a large number of operations simultaneously. Thus, a supercomputer easily works with large data sets and reaches the result by calculating quickly.

UTWCalc programme can calculate the tetration numbers and at the same time take their super logarithm. For this reason, operations with these numbers cannot be calculated on standard computers due to insufficient processor speed. The processing speed of a computer is indicated by the Flops value. The Flops value is obtained by direct mathematical measurement of the performance of a computer. It consists of the initials of the English words FLoating-point Operations Per Second. Flops is divided into Killoflops, Megaflops, Gigaflops, Teraflops, Petaflops, Exaflops. Each flops value is 1000 times higher than the processing power of the previous digit. This value multiplies exponentially as in exponents. For example, 1 Gigaflops is equal to one billion (1.000.000.000.000) Flops. Graph1 shows the flops of a classical computer and a supercomputer used today (Probert, 2013).

While the red box in the graph shows the processing power of a standard computer, the green box belongs to Frontier, the supercomputer with the highest processing power today. Interpreting Graph 1, Frontier has about 1 billion times more processing power than a standard computer.

[14] https://superbilg.itu.edu.tr/ Access date:17.12.2024



Grafik1:Flops değerleri

As a result, calculating the number of tetrations and taking their super logarithm It cannot be done on standard computers due to insufficient processing power. For this reason, the UTWCalc program, which can calculate the super logarithm of tetration numbers, which allows us to perform operations with tetration numbers on standard computers, has been written. The project includes two different inventions with UTW function and UTWCalc programme.

4. Method

The project is a study in the field of pure mathematics and mathematical induction method is used in the project. Mathematical induction is the method of proof used to show that an explicit proposition is true for all positive integers. Thus, during the process "the judgement that is recognised as true for one case is recognised as true for all cases" (Gökdoğan, 2015: 28), provided Mathematical induction is mostly used in computer sciencemathematical logic and . more general terms, it "shows that evaluable (calculable) statements are equivalent.[15] " "The proof of mathematical induction is the proof that is valid for minimal structures, and that given the proposition all if it is valid for direct substructures of a structure S, it must also be valid for S." (Burstall, 1969 as cited in Erarslan, 2024: 13.). In the project, firstly Googlplex numbers were investigated and information was collected from sources related to tetration and other super operation numbers. Then, considering the fact that tetration numbers are repeated exponential operations, general views on the super logarithmic calculation of these numbers were reached.

[15] https://tr.wikipedia.org/ Accessed on 01.01.2024

In the literature review, it was seen that the super logarithm of tetration numbers can be taken, but an exponential operation that can perform repeated logarithm for other super operation numbers has not been developed, and the studies on this field are quite scattered and insufficient.

Therefore, the UTW function which can take the inverses of pentation (hyper operation5), hexation (hyper operation6), heptation (hyper operation7), octation (hyper operation8), ennation (hyper operation9), decation (hyper operation10) was developed. Then, *the functionality of the UTW function was proved on the pentation numbers defined as hyperoperation-5 operations. It is shown that the UTW function found in this study is easily used in all subsets of the operations related to the superoperation numbers and is true for all natural numbers.*

In the software phase of the project, UTWCalc programme was written to take the super logarithm of tetration numbers on standard computers. Waterfall method, which is one of the basic programming methods, was used in the project. "The Waterfall model is a static model and approaches system development in a linear and sequential manner, completing one activity before another" (Fowler, 2004: 429). The sequential stages of the Waterfall model have been defined in various ways by different scholars. According to Pfleeeger and Atlee (2006), these phases consist of seven stages: requirements analysis, system design, programme design, coding, unit and integration testing, system and acceptance testing, operation and maintenance. With the UTWCalc programme, the software phase of which was completed, the number 8 tetration 3 (6.01452075365139E+15151335) was calculated in a short period of 56 seconds without the need for a super computer, and then the super logarithm of this number was finalised in 22 seconds by entering the desired logarithm base and printed on the screen. The programme is written in Java and Python languages. BigInteger, Scanner, os and logging libraries were used in the Java part of the programme and os and logging libraries were used in the Python part. The data type of the programme is BigInteger. BigInteger is a special library and data type created for Java and used in the calculation of large numbers. Scanner library is used to get console entries from the computer. In the Python part of the program, the os library is used to read the data from the hard disk and to call the file where we calculate the number of tetrations before starting the UTW program. The logging library is used to warn the user. For example, when the user calculates the super logarithm, it warns that the number will be infinite when 1 is entered in the base.

5. Findings

5.1. In this section, the working principles of the UTW function will be explained, the UTW function will be proved and the development process and features of the UTWCalc programme will be mentioned. UTW Function Working Principle and Proof of UTW Function

5.1.1. Working Principle of UTW Function

As mentioned before, the super logarithm function is the representation of the inverse of the tetration operation, which is the extreme operator of exponential numbers. Thanks to this representation, the numbers can be minimised by easily performing operations with tetration numbers.

The difference of the UTW function, which is the subject of the project, from the $Slog_x(y)$ function that defines the super logarithm is that it is a function that not only provides the representation of tetration numbers and ease of operation, but also provides the representation of other hyperoperation numbers such as pentation, hexation, heptation, octation, ennation, decation. In the literature review, it was found that super logarithm is a new field of study in pure mathematics and most of the studies are independent studies in the fields of mathematical engineering and computer science [16]. For this reason, most of the studies have been carried out on the inverse of tetration numbers. However, there is no study on the calculation of the inverse of other hyperoperation numbers in the literature.

5.1.2. Proof of UTW Function

UTW is the inverse operation of the PTG function. It works according to certain rules. Just as PTG performs operations on tetration and other hyperoperation numbers (pentation, hexation, heptation, octation, ennation, decation), the UTW function calculates logarithms, superlogarithms and other large numbers (hyperoperation numbers) inverses that the super logarithm function cannot operate on. The rules of the UTW function are as follows;

(I) $Utw_Z(x,y)= {}^{R}_{-(Z+1)}ptg[x,y]$

(II) $Utw_1(x,y)= \log_y(x)$

(III) $Utw_2(x,y)= slog_y(x)$

(IV) $Utw_Z(x,y) = 1+ Utw_Z(Utw_{Z-1}(x,y),y)$, i.e. Utw_{Z-1} is iterated until x <=1, adding 1 at each iteration

(V) $Utw_Z(<=1,y)= 0$

According to Rule (III),

$Utw_2(ptg(16,2),16) = 2$

$Utw_2({}^{5}5,5) = 5$

[16] In computer science, denoted as log*. Performs super logarithms in base 2 only. For detailed information, see Cormen, T., H., Leiserson, C., H., Rivest, R., L. and Stein, C. (2009*). Introduction To Algorithms*(Edition). pp. 56-59. MIT Press.

According to Rule (IV and V)

$$Utw_3({}^{R}_{3}ptg(3,3)3)= 1+Utw_3(Utw_{(2)}({}^{R}_{3}ptg(3,3),3) \qquad \text{(Rule IV)}$$

$$= 1+Utw_3(Slog_3({}^{R}_{3}ptg(3,3),3)= Slog_3(Ptg(\underbrace{ptg(3,3),3)}_{3})+1 = ptg(3,3)+1 =1+Utw_3(3^{27},3)$$

$$= Utw_3({}^{R}_{3}ptg(3,3)3) = 1+1+\underbrace{Utw_3(3^{27},3)}$$

$$1+1+ Utw_3(\underbrace{Slog_3(3^{27})}_{3}) \qquad \text{(Rule V)}$$

$= Utw_3(Slog_3(3))+1+1= 1+1+1= 3$ (The operation is finished since the result will always be added 0 in the next steps).

As a result, the number of steps is 3. 3 pentation step count(3) equals 3 pentation, so we can prove that it works for all positive numbers.

As can be seen from the examples, the UTW function not only inverts the tetration operation, but also inverts the pentation operation and takes its super logarithm. This is also valid for other super operation numbers. Table 5 shows the use of the UTW function in the calculation of other super operation numbers.

Table5: Use of the UTW function in the calculation of other super operation numbers

Operation Rating	**Calculation Method**
Hexation (hyperoperation6)	$Utw_4(x,y)$
Heptation hyperoperation7)	$Utw_5(x,y)$
Octation (hyperoperation8)	$Utw_6(x,y)$
Ennation (hyperoperation9)	$Utw_7(x,y)$
Decation(hyperoperation10)	$Utw_8(x,y)$

5.2. UTW Calc Programme

The UTW Calc Programme, which was developed in the project, is a special programme that allows tetration numbers to be easily calculated on standard computers and at the same time to take the super logarithm. This programme can calculate the maximum ptg[8,3]. In the literature review, it was found that there was no similar work to this programme and that such a large number could not be super-logarithmised on a standard computer.

The programme takes 56 seconds to calculate the number of tetrations. The number ${}^{3}8$ ($8^{16777216}$) has 6.01452075365139E+15151336 digits. The amount of digits in the digits of the calculated number was printed on 2580 A4 size pages. *The logarithm of such a large number by the UTWCalc programme took only 22 seconds.* This is far below the time that a standard computer can calculate. In this respect, the UTWCalc programme is different from other programmes in terms of calculation time.

5.3. UTWCalc (Untowerer Calc) Programme Algorithm Scheme, Code Information and

Interface

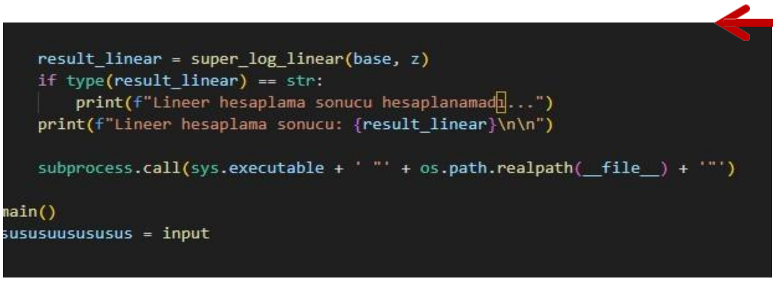
5.3.1. Code Information

The programme loads the BigInteger and Scanner libraries at startup. Then it defines the custompower function (custompower is a function we wrote for exponentiation). Custompower multiplies the number continuously and then decreases the exponent by one until it is zero. Next, it defines a function called Tetration. Tetration does something similar to custompower. The only difference is that it calls custompower repeatedly, not repeated multiplication.

The programme then calls the Scanner library and asks for the base and height numbers.

It then compares the maximum value of the base and height numbers and if it is greater than this, it requests the programme to stop. The programme passes the base and height numbers through the tetration function and prints the number of digits on the screen.

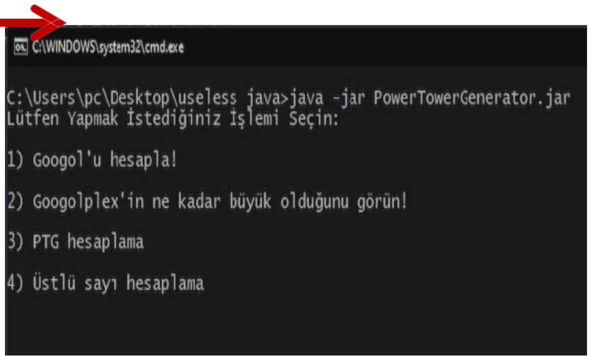
The UTWCalc programme then calculates the number of tetrations. It defines the super_log_linear function to take the super logarithm of the number from its name. The function calculates the super logarithm by iterated logarithm. The programme then defines the main function. The function uses the os and logging library using before tetration reads the number calculated by the programme and then asks the user to enter a number. If the base is one, it prints infinity and ends the process.



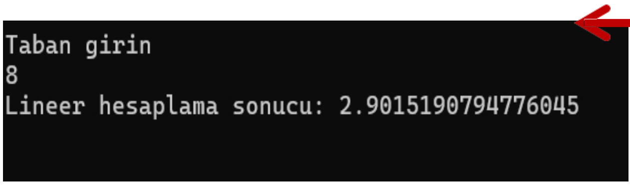
If the base is not 1, the result is calculated with the super_log_linear function and the result is written

5.3.2. Programme Interface and Usage

When the programme is first opened, a calculation menu appears. UTWCalc opens 4 different menus. The menus calculate large numbers such as googology numbers, as well as the PTG menu that calculates tetration numbers and the menu that calculates exponential numbers.



The user can select any of these menus and calculate the number. The UTWCalc programme redirects the user to a second menu to get the super logarithm and a screen appears where the user asks which base they want to get the super logarithm



After entering the base, the result is printed on the screen by taking the super logarithm of the number.

6. Conclusion and Recommendations

20. The most important scientific developments of the century are the discovery of artificial intelligence and space exploration. With these discoveries, mankind has taken a new step discover its journey in the universe. Scientific progress in practical life is realised through mathematics, which offers us a magic wand to analyse nature. The field of pure mathematics paves the way for practical mathematics with its theorems. Thus, each scientific discovery can find its place in life through mathematics. This is a fundamental principle of scientific progress.

This century, which we call the "century of science", is a period in which scientific development has accelerated. The speed of scientific development is directly proportional to how much mathematics is used in practical life. Therefore, we believe that new axioms in pure mathematics will pave the way for future scientific developments.

Operations with large sets of numbers are a new area of pure mathematics. A general By definition, these sets of numbers are Googolplex numbers and are called super operation numbers. The first definition of the Googolplex number was made by Kasner in the early 20th century.

(Kasner et al., 1940: 23). This number is called googolplex number and $10^{10^{100}} = 10^{googol}$ is equal to the number of superoperations. In later periods, interest in the field has increased and super operation numbers the first definition was made by Goodstein. Goodstein defined super operation numbers such as tetration (hyper operation4), pentation (hyper operation5), hexation (hyper operation6), heptation (hyper operation7), octation (hyper operation8), ennation (hyper operation9), decation (hyper operation10) and so on. *Among the super operation numbers, especially tetration numbers have a rapid growth rate sufficient for the requirements of the 21st century. However, considering that space research, quantum physics and particle law studies will increase in the future, it is clear that the functions defined for tetration numbers will not be sufficient. Considering that scientific development is based on mathematical theorems, it is a necessity to develop new functions in this field. For this reason, the UTW function, which can operate on all degrees of superoperation numbers, was developed in the project.* The properties of the UTW function are as follows;

- With the super logarithm notation used today, only the super logarithm of the tetration numbers can be taken from the super operation numbers, while with the UTW function, the inverse of all super operation numbers can be taken.
- Just as multiplication is the inverse of division, addition is the inverse of subtraction, and exponentiation is the inverse of logarithm, the UTW function is the inverse operation of the PTG function we invented earlier.
- The UTW function is a function that allows us to minimise super operation numbers by taking their inverse.
- The UTW function provides convenience in mathematical operations by allowing us to find how many times the super logarithm of the reduced numbers is taken.

In the project, not only a mathematical function (UTW) was developed, but also a UTWCalc program was developed that can easily calculate the tetration numbers easily shown with this function on a standard computer with a standard operating power without the need for a super computer. With the UTWCalc programme, the number ${}^{3}8$ ($8^{16777216}$) *is calculated in 56 seconds* on a standard computer *and its super logarithm is taken in 22 seconds. The calculated number has 6.01452075365139E+15151336 digits and the amount of digits in the digits of the number is printed on 2580 A4 size pages*. UTWCalc is an important software project in terms of the next century's need for calculators with high computational capability and is open for development. The project is considered valuable as it includes two different inventions with UTW function and UTWCalc software. In the last prototype of the UTWCALC program, the software was renewed and a number with ~95,000,000 digits can be calculated in 1 minute 20 seconds, the largest number with all digits calculated.

Googology is one of the new research areas of pure mathematics. The field of study is quite wide. Considering that mathematics studies large sets of numbers such as Googology, the project is expected to make a significant contribution to the popularisation of this field.

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