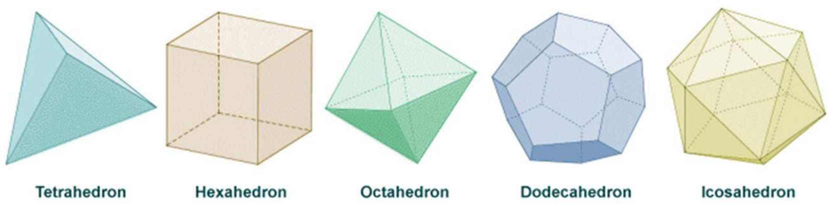
Solid Fun: Why There Are Only 5 Platonic Solids

The ancient Greek philosopher Plato believed that everything around us was made from 4 elements: earth, fire, water and air. Rather obscurely, he associated each element with a 3 dimensional solid. Earth was a cube, fire was a tetrahedron, water was an icosahedron and air was an octahedron. A natural question arises: why these shapes? I, for one, had never heard of an icosahedron before researching this topic, and I can also think of much simpler 3D objects to use instead. The answer is that the four solids above have a special property; they are made of faces which are the all same shape and size, such as the cube which is made of 6 equal squares. A more rigorous definition might also include that they are convex (have no dimples) and their vertices are all the same. Think of them as the 3D version of regular polygons. These so called 'Platonic solids' are particularly interesting as beyond these four, only one other exists: the dodecahedron, which Plato attributed to the heavens. In order to prove that only 5 such solids exist (and perhaps rescue our universe from a devastating new element!) we will first prove a profound result in 3D geometry.



1. The 5 Platonic solids

Proof of Euler's formula for polyhedra

One of the most elegant results the Swiss mathematician Leonhard Euler discovered was his formula for polyhedra. Unlike a lot of higher maths, the proof we are about to cover is mostly manipulation of different shapes, and lacks any equation harder than the formula itself. I hope that the distinct methods we use below will bring you to new ways of thinking, which I believe is the most exciting part of maths. Enjoy!

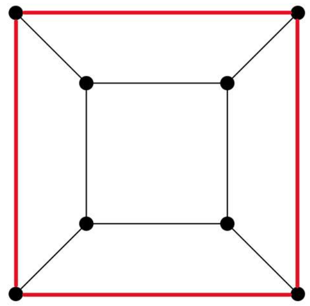
Euler's formula for polyhedra links the number of vertices, edges and faces a polyhedron has. If we use the first letters V, E and F to represent these quantities respectively, it states that:

$$V - E + F = 2$$

To see this in practice, let's use the example of a cube, which has 6 faces, 12 edges and 8 vertices. Substituting in our values of V = 8, E = 12 and F = 6, we see that the left hand side does indeed evaluate to equal 2. Now that we are at least convinced that this formula works, let's consider the proof. To start we consider a hollow polyhedron made from a stretchy material, and remove one of its faces to form a new shape. We'll follow along with a cube to help with visualisation, so this simply becomes what could be thought of as an open box. We may have removed one face, but our new shape still has the same number of vertices and edges. Since a value has changed, the initial formula we had needs changing. Because the left hand side has reduced by one as F is one smaller, the right hand side must also be reduced by one. Our equation for this new shape now becomes:

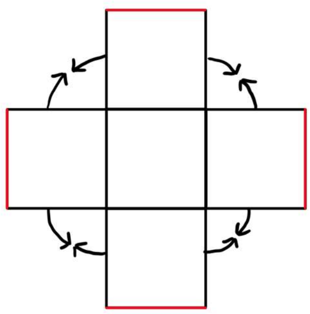
$$V - E + F = 1$$

Now for some tricky spatial thinking. To turn our shape into a 2D representation, imagine pulling the edges of the missing face away from each other until all of the faces of our shape lie flat together. The perimeter of our 2D representation is made from the edges of the missing face. Below is what we would obtain through this process with our open box. The red lines are the edges which were on the face we removed. This also looks like a front on view of our 3D cube with the missing face at the front, and it may be helpful to think of it as this.



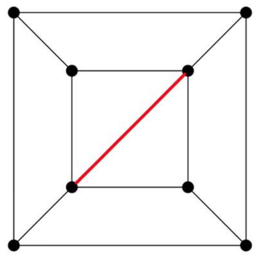
2. The 2D representation of the cube with a removed face

Our process could also be thought of as turning the polygon into a 2D net and then moving the edges so that they border the edge they touch in 3 dimensions. With our open box, this looks like this:



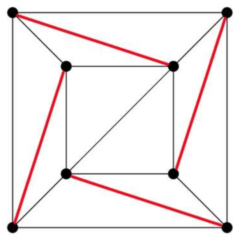
3. The net of the open box

Once we have moved the edges while extending the red sides so there are no new edges introduced, we end up with diagram 2. It's important to note that in this process from 3D to 2D we have produced no new edges, faces or vertices. This means that our modified formula of V-E+F = 1 still applies. Since we are in 2D now, though, we must clarify that a face is just an enclosed shape. Next, we wish to convert all the faces in our 2D diagram into triangles, which are much simpler to work with than other polygons. To do this, consider a face with more than 3 sides and connect two of its vertices to form two smaller faces: a triangle and another polygon. With the central face in our cube as an example, this simply becomes:



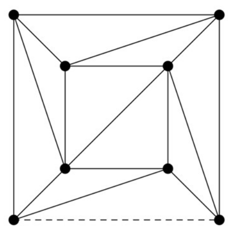
4. Our 2D representation with one face split into two triangles

Crucially, our formula of V-E+F = 1 is still conserved. In drawing a new line we have introduced one more edge, but we have also created one more face. Since we are subtracting E and adding F, these changes cancel out. We now repeat the process of reducing all the faces into triangles until every face is a triangle. With the cube diagram, this becomes:



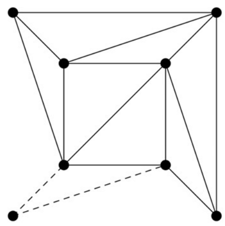
5. Our cube diagram now only with triangular faces

Now the simplicity of these triangles becomes greatly useful. We set up a process to eliminate the triangles one by one to reduce our diagram further until only one face is left, while still conserving V-E+F = 1. If there is a triangle with one edge on the outside of the diagram, we remove that outer edge. This decreases E by 1 but also decreases F by 1 as we have also lost a face. Again, since E is subtracted but F is added in our equation, our equation's value doesn't change. Clever, right? On the cube example, this could look like this:



6. The dotted line is removed, decreasing E and F by one

We also need a way to remove triangles with two outer edges. In this case, we remove both outer edges, as shown below:



7. Two outer edges are removed to eliminate another triangle

This operation also conserves V-E+F = 1. This is because we remove the two outer edges and a face, but also lose one vertex where the two outer edges met. V and F decrease by 1 each, but since E decreases by 2, the changes cancel out in our formula. If we repeat these two steps continuously, we eliminate every face but one, leaving a single triangle. This has 3 vertices, 3 edges and 1 face, quantities which satisfy V-E+F = 1. Bingo! All of the steps in this proof can be reversed, meaning that it is possible to begin with a triangle and end up with a 2D representation of a polyhedron, which fulfils V-E+F = 1. This is important as we did assume V-E+F = 2 initially. Since the 2D representation is missing a face, we have proven that every polyhedron satisfies the formula V-E+F = 2.

Proving there are only 5 Platonic solids

Now we have proven Euler's formula for polyhedra, it may still seem unclear how this can be applied to the Platonic solids. The key idea to use is that each face must be the same on a given Platonic solid, which leads to some relations we can express. First, let us define a new term for a vertex. If a vertex has a degree of x, it means that x edges meet at that vertex. For example, a cube has vertices of degree 3 as three edges meet at them. In a platonic solid, the degree of each vertex must be the same by definition. To relate the number of vertices and edges, we could reason that the number of edges on the solid is the degree of the vertices (x) multiplied by the number of vertices. The only flaw in this is that an edge has a vertex at each end. To fix our overestimation, we simply divide by 2.

$$E = \frac{xV}{2}$$

$$V = \frac{2E}{x}$$

Now to relate the number of faces and edges. Let y be the number of sides each face has, which is constant as all the faces are equal. We could say that the number of edges our Platonic solid has is the number of faces multiplied by the number of edges per face (y). Again, we encounter the same double counting problem from before, as each edge is shared by two faces. Dividing by 2 gives us:

$$E = \frac{yF}{2}$$

$$F = \frac{2E}{y}$$

Since we now have V and F in terms of E, we can substitute these values into V-E+F = 2.

$$\frac{2E}{x} - E + \frac{2E}{y} = 2$$

From here, we can divide both sides by 2E to get:

$$\frac{1}{x} - \frac{1}{2} + \frac{1}{y} = \frac{1}{E}$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2} + \frac{1}{E}$$

Since the number of edges (E) must be positive, 1/E is positive and the left hand side must be strictly greater than ½. Who would have guessed that this entire process would culminate in a simple inequality?

$$\frac{1}{x} + \frac{1}{y} > \frac{1}{2}$$

To solve this inequality, it is important to notice that at least one of x and y must be greater than 4, as if both of them were less than or equal to 4, the left hand side is at most ¼ + ¼ = ½ which isn't greater than the right hand side. This gives us only a few cases to check. We are assisted by the fact that x (the degree of the vertices) must be greater than or equal to 3, as 3 or more edges are required to form a vertex. The same is true for y (the number of edges on a face) as 3 sides is the fewest number possible for a polygon. Being less than 4 and greater than or equal to 3 implies equalling 3. Therefore, when one variable is 3, the other can be 3, 4 or 5 to complete the inequality. This produces 5 pairs of solutions for x and y. From these solutions, we can calculate the values of V, E and F using the relationships we discovered earlier, and find which Platonic solid links to each of our value sets:

x (degree of the vertices)	y (number of sides per face)	Vertices	Edges	Faces	Platonic solid
3	3	4	6	4	Tetrahedron
3	4	8	12	6	Cube
4	3	6	12	8	Octahedron
3	5	20	30	12	Dodecahedron
5	3	12	30	20	Icosahedron

Conclusion

Having gone through such an intensive process of manipulating shapes (and a few numbers) I find it extremely gratifying that the answer we found is so elegant. The symmetry of V and F in the table above truly fascinates me, and the fact that so few Platonic solids exist (and in their respective shapes) is quite unintuitive, making it even more satisfying to prove. Hopefully you have stretched your spatial thinking skills and gained an insight into the techniques used in geometric proofs, and with some luck Plato's theory of the elements won't need modifying any time soon!