Ways to solve and further use the Basel problem.

This essay intends to explore how we can solve the Basel problem further develop it. Exploring the Fourier series route and deriving the Fourier series itself, as well as using Euler's route to derive the Pi Prime product using his approximation of sine curves. These problems are particularly interesting as they deal with sine and cosine graphs in a unique way, making use of their infinite nature to derive some unexpected yet beautifully simple forms. Looking at the two different approaches taken to find the sum of the reciprocal squares, its easy to see how they both share basic similarities, like how the crux of both solutions lies upon the use of a sine wave; however, the further expansion of the outcomes differs wholly. Each solution uses either integration or differentiation, this is one of my favourite parts of the solutions, as they either reduce or increase the powers to arrive at the same conclusion, and it almost seems as if they are both solved in completely opposite directions, but both intersect at the end to get to the right power that gives the solution.

The Basel problem was initially proposed in 1650 by Pietro Mengoli: to find the precise summation of the reciprocal squares for all real numbers. Prior to the 1700's many people were able to find numerical approximations to the Basel problem, just never a precise summation. The difficulty in solving the Basel problem, was that it converges too slowly. There had been many approximations, the longest in 1730 by Moivre who found nice decimal places, and in 1731 Euler unknowingly found eight while working on his solution. The Bernoulli family also attempted this mathematical challenge. The Bernoulli family was one of the most prominent families in mathematics in the early 1700's, but their attempts to solve the Basel problem proved unsuccessful. The family attempted first numerical approximations, and John Bernoulli later consolidated with Gottfried Leibniz, a German mathematician, took the approach of attempting to speed up the convergence of the series, while this led to a mathematical breakthrough, solving the sum of the reciprocal triangular numbers, it did not solve the Basel problem. Attempting to solve the very problem named after their hometown was unfruitful, so, devoid of answers, the world deemed this unsolvable. Euler, also from Basel, sought out the conclusion to this now 84-year-old question, and in extraordinarily little, time had a solution. This question, left unsolved by a family of 8 mathematically gifted children, was the cause for Eulers initial fame.

Euler furthered his work on the Basel problem by using it as the base of his proof for the sum of the prime numbers, the Euler Pi Prime product, which later due to Cauchy Reinmann would be known as $\zeta(2)$. His analytical and algebraic prowess was clearly demonstrated through his summation of the reciprocal square numbers when solving the Basel problem in 1734. Eulers fame is quite an odd one, as his most famous work, Euler's Identity, which relates trigonometric functions and complex numbers, was largely by Roger Cotes rather than Euler. However as, Cotes used geometry to explain his conclusion, it was inaccessible and hard to follow. Cotes's formula was written in the form $\ln(cos(x) + i\sin(x)) = ix$ using natural logarithms. Euler the proved it in an easier and more concise way, claiming it as his own. The Basel problem and his solution is much less well known and yet is his most impressive work. At 28 he had quickly produced one of the best mathematical outcomes of his life, only to be later related to the work of Cotes. In this essay I aim to do Eulers work on the Basel problem justice.

I would like to take this paragraph to explore Euler's work with the Maclaurin series, and the work that began his life as a renowned mathematician. His method of representing sine through two different equations and solving the factors of x^3 to solve the sum of the reciprocal square numbers, and how he used this to continue his work to solve the pi prime product is particularly interesting. Below is the Maclaurin series where Eulers work began:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}(-1)^{n-1}$$

This is one way of writing a sine wave. Euler began by finding a different way to express sine.

He approximated the sine wave as a cubic graph with roots through -π, 0, and π, and because the gradient through (0,0) on a sine wave is 1 he could find an expression as a cubic approximation for sine.

$$y = Ax(x-\pi)(x+\pi) = Ax(x^2-\pi^2) = Ax^3 - Ax\pi^2$$

These are the forms of the approximation for sine where A is a constant to be found, by differentiation:

$$\frac{dy}{dx} = 3Ax^2 - A\pi^2$$

We sub in x=0 and dy/dx=1 $\quad 1 = -A\pi^2$ hence $A = \frac{-1}{\pi^2}$

We can substitute this back into an earlier form of the y equation.

$$y = \frac{-1}{\pi^2}x(x^2-\pi^2) = \frac{-x^3}{\pi^2} + \frac{x\pi^2}{\pi^2} = x - \frac{x^3}{\pi^2}$$

$$y = x\left(1 - \left(\frac{x}{\pi}\right)^2\right)$$

This is the final form for the cubic approximation, but we can make it a close sine approximation and approximate a quintic. The quintic formula is:

$$y = Ax(x+\pi)(x-\pi)(x+2\pi)(x-2\pi) = Ax(x^2-\pi^2)(x^2-4\pi^2) = Ax(x^4 - 5\pi^2x^2 + 4\pi^4)$$

$$y = Ax^5 - Ax^3\pi^2 + 4Ax\pi^4$$

We can find a better approximation now for A using dy/dx=1 at x=0

$$\frac{dy}{dx} = 5Ax^4 - 3Ax^2\pi^2 + 4A\pi^4$$

$$1 = 4A\pi^4 \text{ and hence } A = \frac{1}{4\pi^4}$$

And now substitute our A into the original equation:

$$y = \frac{1}{4\pi^4}x(x^2-\pi^2)(x^2-4\pi^2)$$

Because the whole equation is multiplied, we can divide each bracket by $\frac{-1}{\pi^2}$ *and* $\frac{-1}{4\pi^2}$ (together it is wholly divided by $\frac{1}{4\pi^4}$ when we multiply the brackets together, as we also multiply what we divided by). This means we can re-write the equation as:

$$y = x\left(1 - \left(\frac{x}{\pi}\right)^2\right)\left(1 - \left(\frac{x}{2\pi}\right)^2\right)$$

This quintic approximation shows a pattern. Since we increase the roots to 2π and -2π they are added on the denominator in the bracket. Therefore, since sine is an infinite function it can be written in terms of increasing integers:

$$y = \sin(x) = x\left(1 - \left(\frac{x}{\pi}\right)^2\right)\left(1 - \left(\frac{x}{2\pi}\right)^2\right)\left(1 - \left(\frac{x}{3\pi}\right)^2\right)\left(1 - \left(\frac{x}{4\pi}\right)^2\right)\ldots\left(1 - \left(\frac{x}{n\pi}\right)^2\right)$$

And n→∞ since sine is continuous so it can be written as a product:

$$\sin(x) = x\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n\pi}\right)^2\right)$$

We can make this sin(x) equal to the Maclaurin series and solve it to find the similar numerical powers factors,

$$\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}(-1)^{n-1} = x\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n\pi}\right)^2\right)$$

Which when expressed in its continuous form can be more easily seen.

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\frac{x^n}{n!} = x\left(1-\left(\frac{x}{\pi}\right)^2\right)\left(1-\left(\frac{x}{2\pi}\right)^2\right)\ldots\left(1-\left(\frac{x}{n\pi}\right)^2\right)$$

Here each side has values of x^3 x^5 x^7 x^9 x^{11}, as LHS has only odd numerical powers of x, and RHS can only have multiples of x that go up by a power of 2 each time from 1 so the powers are always odd. This means the factors of each x power must be equal on either side. E.G. $Mx^3 = Lx^3 \; so \; M = L$ and so we can model the factor of each x power.

Via the product rule the values of x^3 can only be obtained by multiplying the x with one bracket and all the other 1's. Hence you would add together all the possible combinations that give x^3

$$\left(x\times-\left(\frac{x}{\pi}\right)^2\times1\times1\times\ldots\times1\right)+\left(x\times1\times-\left(\frac{x}{2\pi}\right)^2\times1\times\ldots\times1\right)+\left(x\times1\times1\times-\left(\frac{x}{3\pi}\right)^2\times\ldots\times1\right)+\cdots+\left(x\times1\times1\times\ldots\times\left(\frac{x}{n\pi}\right)^2\right)$$

This is equivalent to:

$$\frac{x^3}{\pi^2}-\frac{x^3}{4\pi^2}-\frac{x^3}{9\pi^2}-\frac{x^3}{16\pi^2}-\frac{x^3}{25\pi^2}-\cdots-\frac{x^3}{n^2\pi^2}$$

And so if the factors of x^3 are both equal and the x^3 factor for the Maclurin series is $\frac{x^3}{3!}$, then,

$$x^3:\ \frac{-1}{3!}=\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\frac{1}{16\pi^2}-\cdots-\frac{1}{n^2\pi^2}$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\cdots-\frac{1}{n^2}$$

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

This was Eulers route to finding the sum of reciprocal square numbers. This way is particularly accessible as it doesn't involve cosine (or integration) and is a lot more numerical. I like the contrast between this and the Fourier Series, with one using integration and the other differentiation because the premise of the solution is using the same part of maths but in the two different ways, and so it creates this opposing technique to the Fourier series (which we shall next look at) which is really interesting.

Euler also created the Pi Prime Product based of his work with the reciprocal square numbers. By taking out all the even square numbers then factoring out all primes he showed the divergence of prime numbers as we tend to infinity.

We can represent the sum of the reciprocal squares as zeta (2)

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This connection to the zeta function was made through the connection between the hypothesis by Cauchy Riemann and the prime counting function we will explore next. If we multiply each side by $\frac{1}{2^2}$ then we can isolate the sum of the reciprocal even squares:

$$\frac{1}{2^2}\zeta(2) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots + \frac{1}{2n^2}$$

And so we can subtract the even squares from the original function:

$$\left(1 - \frac{1}{2^2}\right)\zeta(2) = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots + \frac{1}{2n - 1^2}$$

$$\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{2^2}\right)\zeta(2) = \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots + \frac{1}{2n - 1^2}$$

So for every $\left(1 - \frac{1}{p^2}\right)$ factored out of the equation, we can take a new prime factor out if:

If $p = prime$ then we can then continuously remove the prime numbers to remove their multiples, and this must continue forever as there are infinitely many primes, and since 2 is a prime number, we can write this removal as a product.

$$\zeta(2) \prod_{p}^{\infty} \left(1 - \left(\frac{1}{p^2}\right)\right) = 1$$

$$\zeta(2) = \frac{1}{\prod_{p}^{\infty} \left(1 - \left(\frac{1}{p^2}\right)\right)}$$

This is Eulers Pi Prime product. It is equal to one as the only number not taken out is 1 and $\frac{1}{1^2} = 1$ so this can become a fraction ($\zeta(2)$) and so is also equal the sum of the reciprocal squares.

A second way of solving the Basel problem, later derived by Fourier, was the Fourier series. Now consider a periodic function repeating every 2π:

$$f(x) = (2 + 2\pi)$$

This allows use to use sin and cosine functions in the expansion of the form sin(x)/cos(x) f(x), as it fits within the periodicity of 2π. Let us generalise f(x) as:

$$f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \ldots + \sin(nx) + b_1 \cos(x) + b_2 \cos(2x) + \cdots + b_n \cos(nx)$$

We can use this equation where a_0 is the y intercept of the function, and we use sine and cosine as they repeat every 2π.

To find the specific values for the constants, we can take the integral of f(x) between π and -π.

As: $\int_{-\pi}^{\pi} \sin(nx)\, dx = 0$ and $\int_{-\pi}^{\pi} \cos(mx)\, dx = 0$ for all multiples of m and n where x=π then all values containing cos or sin can be removed hence leaving:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0\, dx$$

which becomes,

$$\int_{-\pi}^{\pi} f(x)dx = [a_0 x]_{-\pi}^{\pi} = [a_0\pi] — a_0\pi = 2a_0\pi$$

and so we can re-write this as:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$iff\ \ \frac{a_0}{2}\ is\ used\ in\ expanding\ f(x)$$

This is the y-intercept of the graph f(x) we have written through the Fourier series. We can also find an expression for a_n by multiplying the integral of f(x) by sin(nx) so all values will once again be 0, except for $\int_{-\pi}^{\pi} sin^2(nx)$ which is expanded using integration by parts so does not cancel like the others.

We take the integral of:

$$f(x)\sin(nx) = a_0 \sin(nx) + a_1 \sin(x)\sin(nx) + \cdots + a_n sin^2(nx) + b_1 \cos(x)\sin(nx) + \cdots + b_n \cos(nx)\sin\,(nx)$$

Which this simplifies to:

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \int_{-\pi}^{\pi} a_n sin^2(nx)dx$$

And then we can find the LHS:

$$\int_{-\pi}^{\pi} a_n sin^2(nx) = \left[a_n\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\right]_{-\pi}^{\pi} = \left[\frac{a_n x}{2} - \frac{a_n \sin(2x)}{4}\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\pi a_n}{2} - \frac{a_n \sin\,(2\pi)}{4}\right] - \left[\frac{-\pi a_n}{2} - \frac{a_n \sin(-2\pi)}{4}\right] = \left[\frac{\pi a_n}{2}\right] - \left[\frac{-\pi a_n}{2}\right] = \pi a_n$$

We can therefore re-arrange this back in the LHS to give a single value for a_n .

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \pi a_n$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

To find a value for b_n we can employ the same strategy using cos(nx) to isolate $b_n cos^2(nx)$. So, we will multiply all values in f(x) by cos(nx).

$$f(x)\cos(nx) = a_0 \cos(nx) + a_1 \sin(x)\cos(nx) + \cdots + a_n sin(nx)\cos\,(nx) + b_1 \cos(x)\cos(nx) + \cdots + b_n \cos^2(nx)$$

We then take the integral of either side through π and -π, which again simplifies as all other integrals will be equal to 0.

$$\int_{-\pi}^{\pi} f(x)\cos(nx) = \int_{-\pi}^{\pi} b_n cos^2(nx)$$

Taking the LHS:

$$\int_{-\pi}^{\pi} b_n cos^2(nx) = \left[b_n\left(\frac{x}{2} - \cos(2x)\right)\right]_{-\pi}^{\pi} = \left[\frac{b_n x}{2} - b_n\cos(2x)\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\pi b_n}{2} - b_n \cos\,(2\pi)\right] - \left[\frac{-\pi b_n}{2} - b_n\cos(2\pi)\right] = \pi b_n$$

Here we can put our simplified LHS into the original equation and find the specific value for b_n.

$$\pi b_n = \int_{-\pi}^{\pi} f(x)\cos(nx)$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\,(nx)$$

After finding the values of a_0 , a_n and b_n we can re-write the series f(x) in terms of our constants. We take the sum of the sine and cosine functions through n because the graphs continue forever, we have already found the constants in terms of f(x).

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=1}^{\infty} b_n\cos\,(nx)$$

This is how the Fourier Series was derived. We can see through this working that the sine and cosine laws are used simply because they represent two functions with a period of 2π and so the deriving of this series from these two functions now shows a series that can be used to map any periodic function through 2π. The Fourier series can be used to map waves and functions, which is a vastly different outcome from Eulers work.

Now, using this formula, we can prove the sum of the reciprocal squares by multiplying either side through by F(x). This is the solution to the sum through the Fourier series using integration by parts. However we need to put all the values a_0 a_n b_n to have the same factor, so we will use the new formula:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, sin(nx) + \sum_{n=1}^{\infty} b_n cos\,(nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

Now we can us this formula to find the sum of the reciprocal squares. We do this by multiplying each side through by f(x) to give the equation:

$$(f(x))^2 = \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty} a_n f(x)\, sin(nx) + \sum_{n=1}^{\infty} b_n f(x) cos\,(nx)$$

To find an equation for the sum of the reciprocal squares we can again take the integral of each side through π and -π and to simplify the equation we can divide either side through by π, giving:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{a_0}{2}f(x)dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n f(x)\sin(nx)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n f(x)\cos(nx)\,dx$$

We can take the values: $a_0,\ a_n, and\ b_n$ out of the integrals as they act as constants in the equation and can therefore be factored out:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2 dx = \frac{a_0}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx\right) + \sum_{n=1}^{\infty} a_n\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx\right) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx\right)$$

We can see that in the brackets next to the values $a_0,\ a_n, and\ b_n$ is the formula for $a_0,\ a_n, and\ b_n$ respectively. We essentially squared each side as they were both equal to f(x) but now we can write this equation in terms of these 3 values:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2 dx = \frac{a_0}{2} + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \times \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \times \sum_{n=}^{\infty} b_n$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2 dx = a_0 + \sum_{n=1}^{\infty}\left({a_n}^2 + {b_n}^2\right)$$

This is known as Parseval's Theorem, which was originally formed to give the total energy of a signal as it is equal to the total energy computed in a frequency domain, and so relates to physics, auto communications and audio processing. To find the sum of the reciprocal squares we can sub in the value f(x)=x to get n^2 through the $\left({a_n}^2 + {b_n}^2\right)$ and then we can isolate the n^2 to give the square numbers, n^2 is always a square number since n is always an integer, so if we find the sum of n^2 we can find an expression for the sum of the reciprocal squares. By subbing f(x)=x into $a_0\ \ b_n\ \ a_n$ we can find their values:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\left[\frac{\pi^2}{2}\right] - \left[\frac{(-\pi)^2}{2}\right]\right) = \frac{1}{\pi}\left(\left[\frac{\pi^2}{2}\right] - \left[\frac{\pi^2}{2}\right]\right) = 0$$

Next we can find the value for a_n, then square the result and find the sum:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\left[\frac{-\pi\cos(n\pi)}{n} - \frac{\sin(n\pi)}{n^2}\right] - \left[\frac{\pi\cos(-n\pi)}{n} - \frac{\sin(-n\pi)}{n^2}\right]\right) = \frac{1}{\pi}\left(\frac{-\pi\cos(n\pi)}{n} - \frac{\pi\cos(n\pi)}{n}\right)$$

$$= \frac{1}{\pi}\left(-\frac{2\pi\cos\,(n\pi)}{n}\right) = \frac{-2\cos\,(n\pi)}{n} = \frac{(-2)(-1)^n}{n}$$

Because cosine is periodic through π at 1 and -1 if we substitute cos(nπ) for $(-1)^2$ it changes the sign depending on if it is an odd or even number of π. Cosine of an odd number of π is negative and an even number of π is positive. So, we can write it at a power of n to -1.

Finally:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx = \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos\,(nx)}{n}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\left[\frac{\cos\,(nx)}{n}\right] - \left[\frac{\cos\,(-nx)}{n}\right]\right) = 0$$

Again as sin(nπ)=0 always it is cancelled out of the equation and as cosine is reflected in the y axis cos(nπ) and cos(-nπ) are both equal. So we can now substitute these values back into our final equation:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2 dx = 0 + \sum_{n=1}^{\infty}\left(\left[\frac{(-2)(-1)^n}{n}\right]^2 + 0^2\right)$$

Where we can simplify the RHS:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\left[\frac{\pi^3}{3}\right] - \left[\frac{-\pi^3}{3}\right]\right) = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

Hence we can substitute this into the equation above it to find the final form of our formula:

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\left(\frac{(-2)(-1)^n}{n}\right)^2 = \sum_{n=1}^{\infty}\frac{4}{n^2} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

This simplifies if we divide each side by 4 to the sum of the reciprocal square numbers:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This proof is the route taken by the Fourier series to give a formula for the sum of the reciprocal squares. This is based upon the sine and cosine graphs being periodic every 2π. The interesting differences between methods should now be obvious. This expansion involves integrating through π and -π to remove the sine and cosine functions and so the main series used is actually removed in the final equation.

In conclusion these pieces of work are so interesting because although these pieces are interlinked by their connection to the sine wave, their further outcomes past the sum of the reciprocal squares are vastly different. This is what I think is the most important part of maths. How a single solution solved from multiple different processes can result in different and equally as important outcomes, and how different the processes can be to reach the same point, showing how varied and diverse maths is. I really love how, as you have now seen, one solution uses integration and the other differentiation, to me it almost seems one worked below the answer and integrated to get to the correct power, while the other worked above the answer and differentiated to get down to the correct power. It shows how, while there are no direct roots to a solution, we can work around the question to later get the answer. I hope this now gives you a sense of appreciation for Eulers earlier work and also gives you some insight into Fourier's way of solving the Basel problem. Most of all I hope this essay has proved to you that while maths is no mean feat it can be beautiful and unique.