The Lemniscate of Bernoulli: A Bridge Between Algebra and Geometry

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1 Introduction

Imbued throughout history, mathematics has created a bridge between subjects that may initially seem disconnected from one another, while simultaneously expanding our understanding of the universe itself. One of those connections is the lemniscate of Bernoulli, a figure-eight shape that creates a connection between algebra and geometry [1]. Jacob Bernoulli examined it in the late 1600s, where he looked at curves related to those of an ellipse and a hyperbola [2]. It appears to be structured similarly to an ellipse; however, it introduces an entirely new setting in mathematics distinct from standard conic sections [3]. The equation based on the hyperbola further develops its connection in algebra to functions [4]. The geometric connection enables the production of even more complex objects, such as elliptic integrals and complex analysis [5]. The lemniscate's unique geometric and algebraic properties have led to significant developments in elliptic integrals and complex analysis, demonstrating its ongoing importance in modern mathematical research. [6].

2 Hyperbola

A hyperbola is a set of points in a plane whose distances from two fixed points in a plane have a constant difference. The two fixed points are the foci of the hyperbola.

If the foci are F_1 (-c, 0) and F_2 (c ,0) and the constant difference is 2a, then a point (x,y) lies on the hyperbola, if and only if:

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a \quad \text{Eq. (1)}$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining:

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \quad \text{Eq. (2)}$$

By looking at this equation, you can assume it to be one of an eclipse, that being:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{Eq. (3)}$$

But now, $a^2 - c^2$ is negative because 2a, being the difference between two triangles PF_1F_2 is less than 2c, the third side. Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown in the diagram below, $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$.

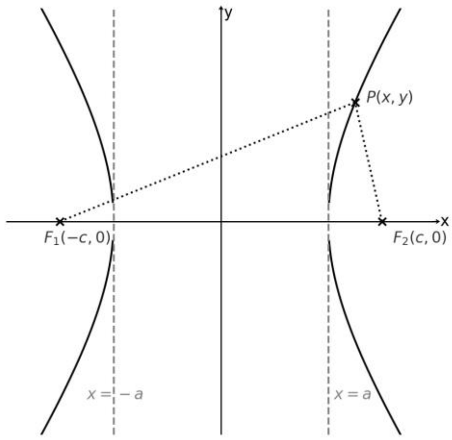


Fig.1

The algebraic steps leading to Eq. (2) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < a < c$ also satisfies Eq. (1). A point therefore lies on the hyperbola if and only if its coordinates satisfy Eq. (2). If we let b denote the positive square root of $c^2 - a^2$, then b will be:

$$b = \sqrt{c^2 - a^2} \quad \text{Eq. (4)}$$

Then $a^2 - c^2 = -b^2$ and Eq. (2) takes a more compact form, this being:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ Eq. (5)}$$

The differences between Eq. (5) and the equation for an eclipse, Eq. (3), are the minus sign and the new relation:

$$c^2 = a^2 + b^2 \quad \text{From Eq. (4)}$$

Like the ellipse, the hyperbola is symmetric with respect to the origin and coordinate axes. It crosses the x-axis at the points (±a, 0). The tangents at these points are vertical because

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} \quad \text{Obtained from Eq. (5) by implicit differentiation}$$

is infinite when y = 0. The hyperbola has no y intercept; in fact, no part of the curve lies between the lines x = -a and x = a. [7]

3 A Lemniscate

A lemniscate is a type of curve that resembles a figure-eight. The name comes from the Latin word 'lemniscatus', meaning "ribboned," for obvious reasons, considering its shape. There are different ways to define this class of curves algebraically and geometrically, as these curves appear in different specialised areas, such areas often connected to the study of elliptic functions and complex analysis, but also significant in fields such as dynamical systems and orbital mechanics. A general form of a lemniscate can be written in Cartesian coordinates as:

$$(x^2 + y^2)^2 = a^2(x^2 - y^2) \quad \text{Eq. (6) [9]}$$

In this equation, x and y are the Cartesian coordinates and a is a constant that determines the size and scale of the curve. This equation describes a closed, symmetric curve in the plane. The lemniscate consists of two lobes that meet at the origin, forming a figure-eight shape. It is symmetric with respect to both the x-axis and y-axis.

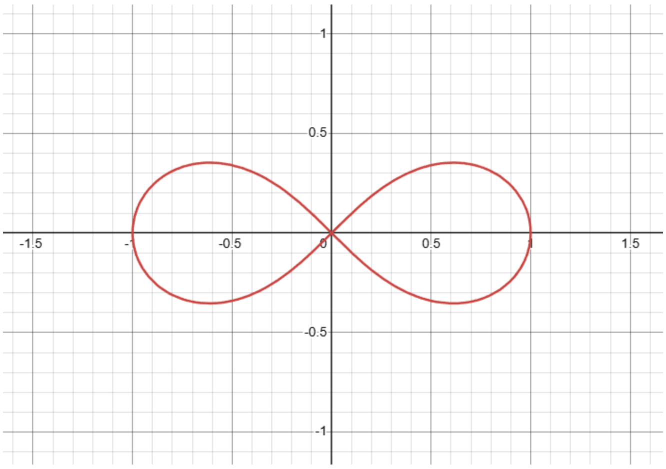


Fig.2 (Where a = 1 in Eq. (6)(Desmos))

The lemniscate represents a special shape that creates a closed circular loop in a geometric space rather than extending forever like hyperbolas or parabolas, which can be very useful in areas such as dynamical systems and physical modelling. [10]. Moreover, the lemniscate possesses two sharp points located at the origin (0, 0), as both curve sections intersect through the origin. The sharp meeting point of two curve branches in a cusp makes the tangent line non-existent because there is no clear direction. A lemniscate possesses solution sets consisting of real and complex values. The equation of a lemniscate has both real and complex solutions. In real terms, it describes the intersection points of the curve, while the complex solutions extend the curve into a broader, multidimensional context, which is linked to complex analysis [11].

An inversion using a circle of radius R with the centre located at the origin inverts a hyperbola into a lemniscate. More precisely, when a rectangular hyperbola with its two asymptotes perpendicular to each other is subjected to this inversion, the two sides of the hyperbola reconnect, and it creates a figure eight, the shape of a lemniscate. This is shown below.

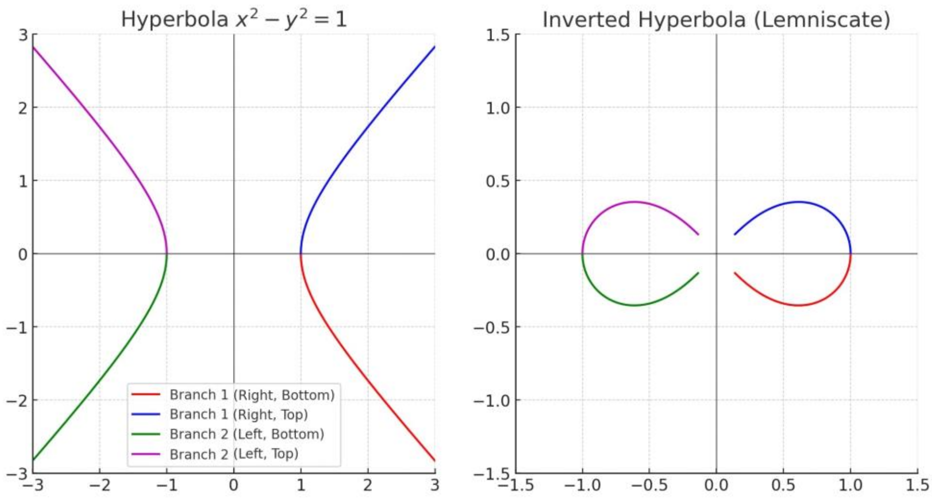


Fig.3 (Where a and b = 1 in Eq. (5) and an exaggerated gap)

Now you might ask yourself, why is there a gap in the middle of the Lemniscate? A cusp appears when a curve shows a sudden bend which lacks clear tangent direction at the singularity point. The polar coordinates representation of a lemniscate of Bernoulli curve utilizes this equation:

$$r^2 = a^2 \cos(2\theta) \quad \text{Eq. (7)}$$

The equation generates a double-looped shape which gets close to the origin (where r = 0) without intersecting at any point. At the origin the lemniscate demonstrates two separate cusp points which are sharp points. The distinct segment between the lemniscate's branches forms the central empty area.

When approaching the origin, we take the limit of r approaching 0, that being:

$$\lim_{r \to 0}(r^2 = a^2 \cos(2\theta))$$

The curve shows no smooth point of intersection as it moves toward each other from opposing directions. The function Eq. (7) prevents the radius r from being zero for both branches at the same time except through discontinuities, so a sharp directional change occurs. The two branches never touch the origin because they remain at a non-zero distance during their approach. At the juncture of a cusp curve there exists no tangent because the curve does not have a well-defined derivative at this location. A smooth connection with intersection occurs at different points from this point in the curve. A tangent describes a straight line which touches the curve only at a single point without passing through the curve section. A cusp prevents the establishment of a tangent line because its steep curve destroys the ability to draw such lines at the origin point. The rapid shift in direction at this section creates an existing void for a well-defined tangent line and thus contributes to the middle gap. A gap appears because it is the result of the singular nature of the lemniscate at the origin. The gap found in physical systems modelled by lemniscates leads to significant real-world effects which include dynamical system trajectories and models of orbital mechanics when lemniscates emerge. The gap in the middle of the lemniscate arises due to the cusp singularities at the origin. The sharp meeting of the two branches without a smooth transition or defined tangent at the origin results in a small, nonzero distance between the branches, creating the gap. This is a fundamental property of many curves with cusp singularities, and it distinguishes the lemniscate from other closed curves, like circles or ellipses, where the curve smoothly meets itself without such a gap. [12] [13]

4 The Lemniscate of Bernoulli

Jakob Bernoulli from Switzerland first introduced the Lemniscate of Bernoulli as an important mathematical curve during his description in 1694. Bernoulli created the curve by adapting the ellipse so that he could research its individual characteristics and practical applications. Bayoli showed no knowledge that his curve was a member of the Cassini oval family, which Giovanni Cassini first studied in 1680. Mathematicians were drawn to study the distinctive figure-eight shape of the lemniscate since its introduction because of its intriguing nature, successfully sparking further investigations from prominent figures, including Giovanni Fagnano and Leonhard Euler. In 1750, Fagnano investigated the fundamental features of the lemniscate, until Euler added to the theory through his 1751 research about the curve's arc length in the mathematical study of elliptic functions [14]. Where does the Lemniscate of Bernoulli derive from? Despite the standard equation of a hyperbola being Eq. (5), a standard rectangular hyperbola has equal semi-axis (a=b), meaning its equation simplifies to:

$$x^2 - y^2 = c^2$$

Or equivalently,

$$xy = c^2 \quad \text{Eq. (8)}$$

The second form, Eq. (8), is a more useful way to express a rectangular hyperbola rotated by 45°. This form arises in natural coordinate systems when dealing with inverse transformations. Now let's have Eq. (8), where c is constant and apply inversion with respect to the origin using circle of radius R. Note, (x,y) are the original coordinates before the inversion and (x',y') are the new coordinates after the inversion. Using inversion about the origin with radius R, the transformation equations are:

$$x' = \frac{R^2 x}{x^2+y^2}, y' = \frac{R^2 y}{x^2+y^2} \quad [9]$$

Rearranging for x and y gives:

$$x = \frac{x'(x'^2+y'^2)}{R^2}, y = \frac{y'(x'^2+y'^2)}{R^2}$$

Substituting these into the hyperbola Eq. (6):

$$\left(\frac{x'(x'^2+y'^2)}{R^2}\right) \cdot \left(\frac{y'(x'^2+y'^2)}{R^2}\right) = c^2$$

Simplifying:

$$\frac{x'y'(x'^2 + y'^2)^2}{R^4} = c^2$$

Multiplying both sides by R^4:

$$x'y'(x'^2 + y'^2)^2 = c^2R^4$$

Rearranging the equation:

$$(x'^2 + y'^2)^2 = \frac{c^2R^4}{x'y'}$$

To match the form of the Bernoulli Lemniscate, we use the property that in an inversion, the product $x'y'$ transforms symmetrically into a difference expression. For a standard inversion where R = 1, this simplifies to:

$$(x'^2 + y'^2)^2 = 2c^2(x'^2 - y'^2) \quad \text{Eq. (9)}$$

Which is the standard equation of the Bernoulli Lemniscate.

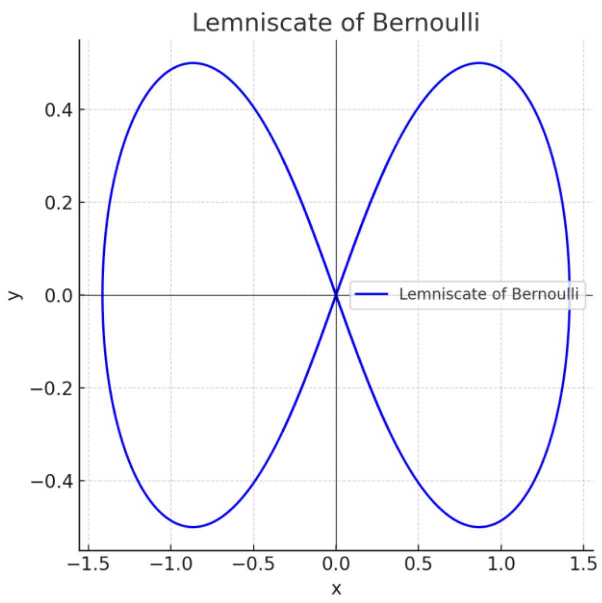


Fig.4 (Lemniscate of Bernoulli graphed where c = 1 in Eq. (9))

Mathematical fields consider the Lemniscate of Bernoulli to be highly significant. Research on the lemniscate led to the creation of elliptic functions, which expand trigonometry and serve number theory and physics. Mathematicians encountered elliptic integrals through their work on determining the lemniscate's arc length, as these integrals are essential for computing the arc lengths of ellipses and their associated curves. The distinctive geometric traits of the lemniscate function have motivated engineering designers and artists to incorporate its shapes into their work for aesthetic quality and strong performance features. Due to both its infinite character and visually pleasing symmetry, the curve has become a focus of interest for people outside of pure mathematical fields [15].

5 The Lemniscate of Bernoulli in 3D

The classical Lemniscate of Bernoulli exists as a two-dimensional mathematical shape, which produces its outline through the relation described in Eq. (9). The extension of the curve into three dimensions adds advanced mathematical characteristics. By defining the z-axis as:

$$z = (x^2 + y^2)^2 - 2(x^2 - y^2) \text{ Eq. (10)}$$

The 3D surface derived from this definition maintains the original lemniscate symmetry yet adds new geometric features, along with analytical traits. This is seen bellow.

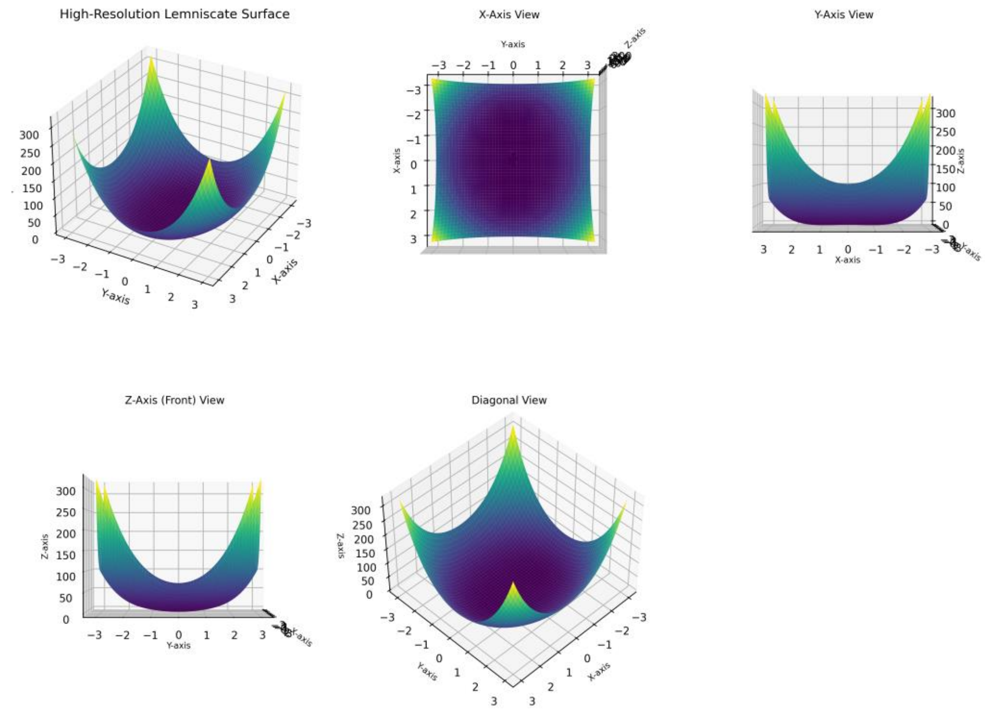


Fig.5

3D representation of the Lemniscate of Bernoulli where the z axis is defined as Eq. (10)

The z-axis rotational symmetry of the surface leads to the determination of its critical points through the solution of:

$$\nabla z = 0$$

When the equation evaluates its points, both extreme local values and saddle points emerge to show essential properties of surface curvature and topology [1]. The Gaussian curvature represents important aspects of the local geometric structure of this surface. The Gaussian curvature reveals the local topological features, which include elliptic, hyperbolic, and parabolic characteristics of the surface area [2]. The recent introduction of 3D lemniscate innovation leads to special singularities, which hold important roles for both algebraic geometry research and singularity theoretical work [3]. The higher-order polynomials in the definition of z create regions where tangents fail to define points and lead to singular points. The xy-plane projection identifies points that correspond to dual mathematical solutions, which produce self-crossing curves. The identified points within the final projection identify solutions from higher-degree polynomial equations as described in complex surface analysis of algebraic geometry. Bifurcations and catastrophe theory allow scientists to establish classification schemes, which identify singular point local behaviours. The assessment of cusp, node, and higher-order singular conditions can be performed through Hessian matrix calculations of z since these three conditions influence surface global structures [5].

6 The Lemniscate of Bernoulli in 4D

4D Lemniscate Orbit around Two Black Holes with Vertical Motion [18]

Having a lemniscate merely graphed in 4d would be extremely impractical and nearly impossible to represent. Instead, the 4th dimension used here is the gravitational force, where the lighter the colour, the stronger the force. Hypothetically, such an orbital arrangement for a planet moving in a lemniscate path near two black holes represents an extremely improbable combination of precise environmental factors, yet it is still possible. Two black holes should exist together in an equal mass configuration, while a low-mass test particle planet functions within a restricted three-body problem scenario. The precise setup of initial parameters makes it possible for a planet to execute periodic motion as a figure-eight between two black holes. Extreme orbital instability would plague this system because gravitational perturbations, coupled with relativistic effects, along with the slow release of gravitational waves, would eventually reshape the system. Numerical solutions of Newtonian mechanics show that figure-eight orbits exist as theoretical constructs, but their maintenance in astrophysical systems proves to be impossible, making them purely theoretical, rather than true astrophysical occurrences.

However, a more accurate representation of an unstable orbit between two blackholes would look more like this: Realistic 3D Orbit around Two Black Holes. [18] In this example, the darker the colour, so dark blue, is the start of the orbit. Yet this is still highly theoretical in the regards that the blackholes are stationary and their gravitational fields do not interact with one another.

Lemniscate illustrations in a figure-eight configuration provide researchers in science and mathematics with a method to study complex gravitational systems while analysing extraordinary orbital outcomes. The problem contains specific instances of three-body solutions representing objects that precisely maintain their motions between larger bodies. Such research into gravitational shapes helps scientists study phenomena and examine the current boundaries of physics while constructing theoretical models to explain cosmic black hole processes. The analysed orbital patterns serve a dual purpose in space exploration by enabling spacecraft navigation efficiency, which decreases operational fuel demands. Scientists employ lemniscate orbits as test models through symmetry labelling because these orbits serve both educational purposes and provide computational spaces, as well as enabling original resonance research. Researchers gain an understanding of cosmic motion fundamentals through studying lemniscate orbits, despite their improbable rarity in space.

7 5D Kaluza-Klein Theory

The 5D Kaluza-Klein theory extends relativity by adding a fifth dimension, where gravity and electromagnetism emerge from spacetime geometry, with the extra dimension compactified. 5D Kaluza-Klein theory and the lemniscate of Bernoulli are connected through essential links between extra-dimensional structures and their association with symmetries across multiple dimensions. General relativity expanded through Kaluza-Klein theory, which integrated a fifth spatial dimension that transformed electromagnetic phenomena into geometrical properties of expanded spatial space. This theory requires physical objects to enter periodic motion within its extra dimension, which produces detectable changes to their measurable properties within standard four-dimensional space-time. The figure-eight curve of Bernoulli emerges in numerous physical and mathematical contexts to depict harmonic balance as well as chaotic systems and orbital mechanics and phase space representations. The shape reaches its stable but never static equilibrium state through mechanisms involving forces and symmetries from potential systems that control its movement patterns. The basic part of dimensional-bound circular patterns exists in both perspectives because KK theory demonstrates the movement of particles in fifth-dimensional loops similarly to the classic mechanics relationship between lemniscates and periodic exchanges of energy and momentum. A KK theory exhibits its extra-dimensional cycle as a visible depiction which parallels lemniscate movement observed within orbital mechanics. Generalised relativistic mechanics generates complex orbital solutions as a result of non-trivial spacetime curvature, and certain solutions exhibit lemniscate-like structures when projected into lower-dimensional space. 5D geodesics establish the definitions of minimum length paths through curved spacetime while preserving their mathematical connections to lemniscates. During its passage through compactified dimensions, a particle generates repeating lemniscate patterns in four-dimensional space due to the mechanical connections between multiple dimensions. The theory shows that theoretical extra dimensions lead to observable space motion patterns through repeated shapes in lower-dimensional space. [16] [17]

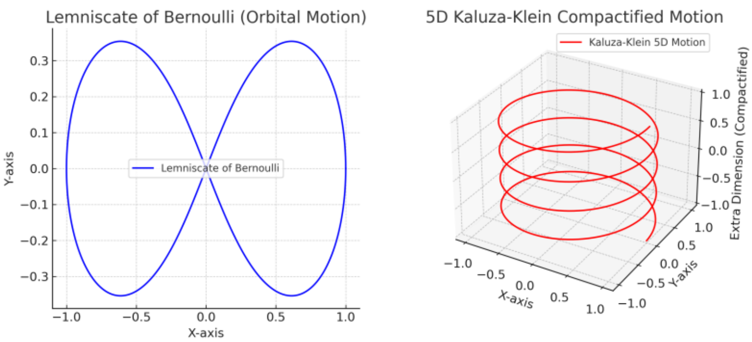


Fig.6

This image demonstrates the connection between the Lemniscate of Bernoulli and 5D Kaluza-Klein theory. The left part of the image demonstrates the lemniscate shape for orbital periodicity, while the right part illustrates a 5-dimensional helical movement, explaining the compactified extra-dimensional particle path from KK theory. Common cyclic patterns observed in the lemniscate allow the shape to serve as a mathematical representation of the 5D Kaluza-Klein theory, providing evidence that higher-dimensional physical processes potentially create lemniscate orbits that link orbital mechanics and theories with extra dimensions.

8 Lemniscates, Fractals and Iterations

The Bernoulli lemniscate in Cartesian form is Eq. (9), where c is the scaling constant. We now need to change this into complex notation, that being when:

$$z = x + iy$$

And the complex conjugate of z being:

$$\bar{z} = x - iy$$

This then makes the magnitude squared of z to be:

$$\bar{z}z = (x + iy)(x - iy) = x^2 + y^2$$

Thus, the left-hand side of the original equation,

$$(x^2 + y^2)^2$$

Becomes:

$$(z\bar{z})^2$$

The right-hand side of the equation is:

$$2a^2(x^2 - y^2)$$

From complex numbers we know that:

$$x^2 - y^2 = Re(z^2)$$

Because:

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

The real part of z^2 is just $x^2 - y^2$, so:

$$Re(z^2) = x^2 - y^2$$

Thus, the right-hand side of the equation,

$$2c^2(x^2 - y^2)$$

Becomes:

$$2c^2 Re(z^2)$$

Now we substitute everything back to get the complex version of the lemniscate of Bernoulli, that being:

$$(z\bar{z})^2 = 2c^2 Re(z^2)$$

Using this, the iterative system used to generate the Lemniscate of Bernoulli fractal is typically based on a complex quadratic map, which can be described as:

$$z_{n+1} = 2c^2 Re(z_n^2) z_n$$

Using this, we can now form iterative graphs, these are shown below:

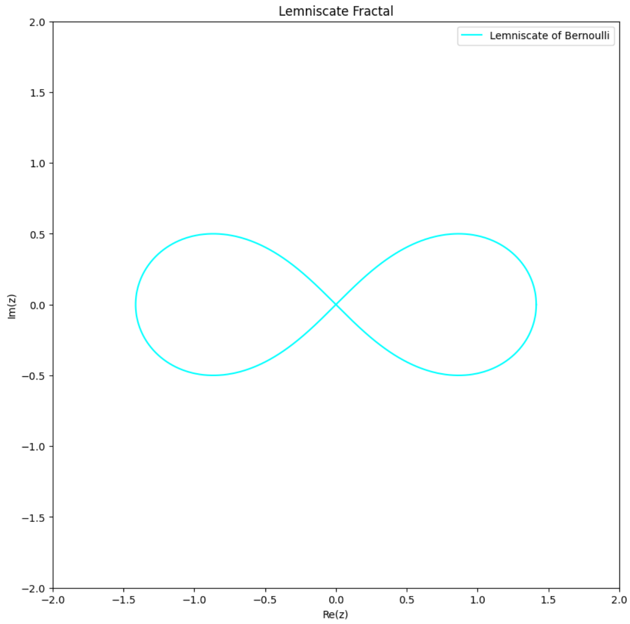


Fig.7 (0 iterations)

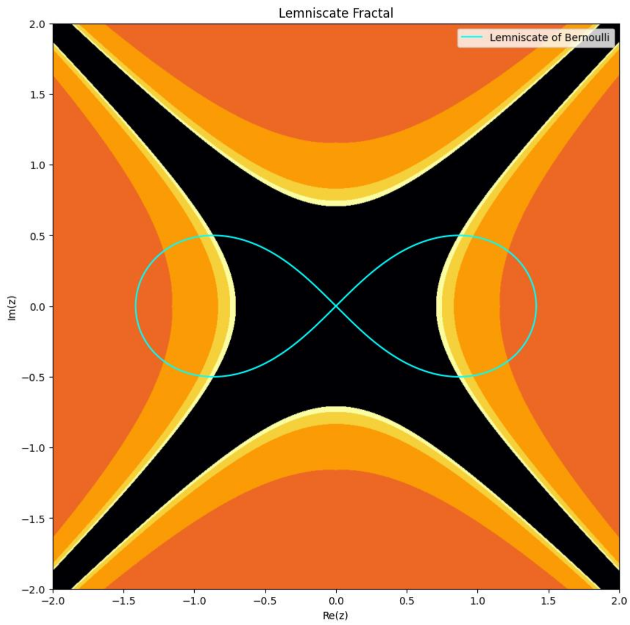


Fig.8 (10 iterations)

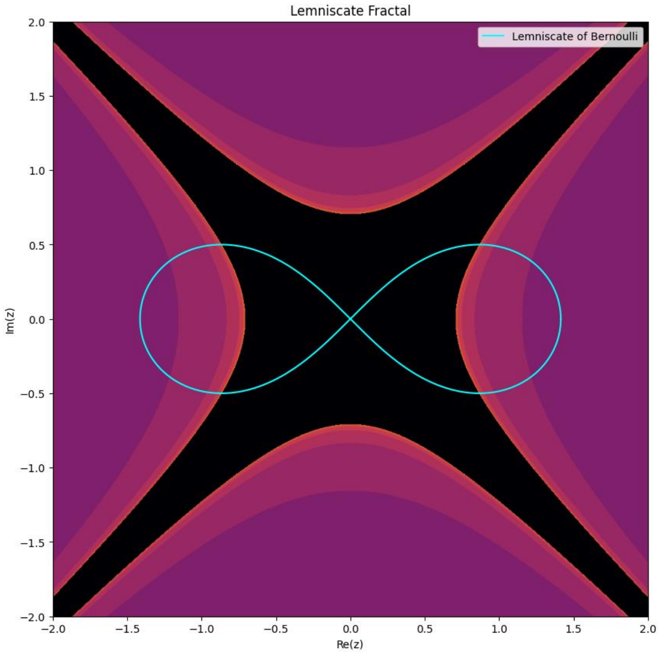
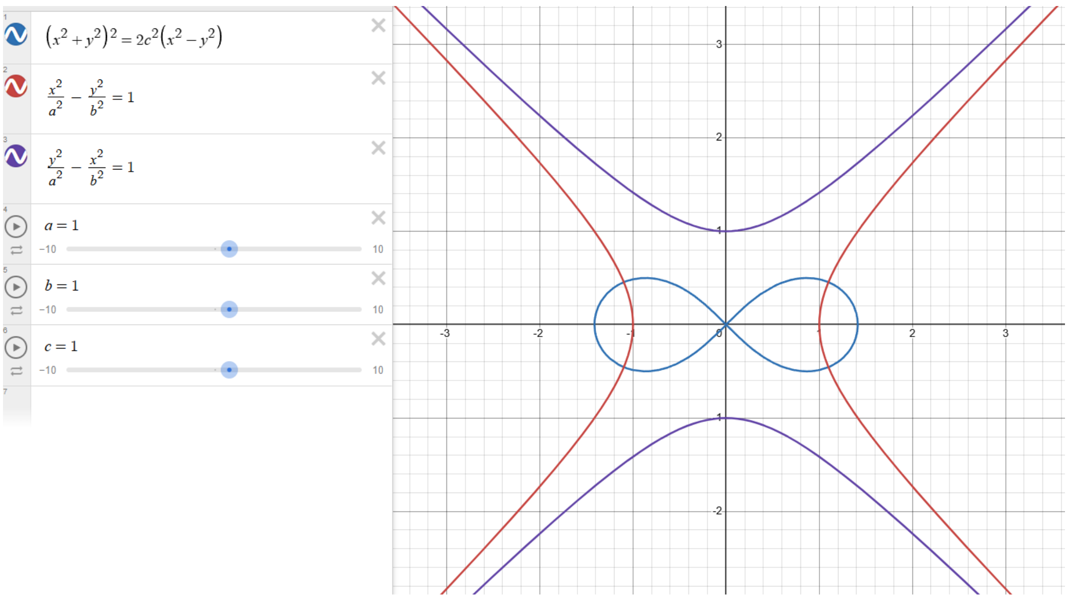


Fig.9 (100 iterations)

The darker colours represent the points that took more iterations to escape. The brighter colours represent points that escaped quickly, so less iterations to escape. This iteration produces a fractal pattern that reflects the geometry of the Lemniscate of Bernoulli, with two hyperbolic branches emerging from the behaviour of the system. This is seen bellow:



Through unexpected results like these, the beauty of mathematics really comes to life. The Lemniscate of Bernoulli is a fascinating mathematical object, and its fractal iterations reveal unexpected structures—namely, hyperbolas—showing how simple iterative processes can lead to rich, complex patterns. It links algebraic geometry, dynamical systems, and complex analysis, offering insights into how mathematical objects evolve under iteration.

9 Conclusion

The Lemniscate of Bernoulli exists as an outstanding mathematical curve that connects algebraic properties to geometrical forms, providing an extensive understanding across diverse mathematical and physical disciplines. Ever since Jacob Bernoulli introduced it in the 17th century, the lemniscate has remained a powerful force that generates contemporary mathematical discoveries through complex analysis, dynamical systems, and higher-dimensional theories. The hyperbola-originated lemniscate reaches its unique form through inversion transformations, which reveals complex connections among mathematical elements. The lemniscate showcases versatility in modelling real-world and abstract phenomena by being studied in 2D, 3D, as well as theoretical 4D and 5D contexts, and revealing its properties for modelling gravitational interactions and extra-dimensional physics. The iterative fractal-generation processes illustrate that basic algebraic structures can produce sophisticated self-replicating patterns, thus demonstrating the important role of the lemniscate in studies between chaos theory and computational mathematics. Beyond its curvature, the lemniscate demonstrates itself as a vital mathematical connection that enables new discoveries to emerge in many scientific fields. Its singular properties and profound mathematical underpinnings protect the lemniscate as a topic that will sustain mathematicians, physical scientists, and theoretical investigators committed to exploring the frontiers of mathematical science.

10 Extras

Why did I choose such a niche topic? As someone who relishes the challenges of MAT questions on a regular basis, I came across question 4 part vi on the 2022 MAT where the answer really intrigued me. The question was:

"Point A is at $(-1, 0)$ and point B is at $(1, 0)$. Curve C is defined to be all points P that $|AP| \times |BP| = 1$, that is; the distance from P to A, multiplied by the distance from P to B, is 1.

(vi) Sketch the curve C, including any parts of the curve with $x < 0$ or $y < 0$ or both."

The answer resembled the graph of a lemniscate. Due to its fascinating appearance, I decided to further investigate the origins of what truly made such a mathematically perfect shape. The Tom Rocks Math Essay Competition seemed the perfect opportunity to showcase my interest for this shape - the Lemniscate of Bernoulli.

I took inspiration from the layout of last year's over 18s winner, which was about 'Clepsydras, Torricelli's Law and Gabriel's Horn', whereby Flynn Nugent had shown a deep interest and had investigated in depth a niche topic, similar to what I have done myself.

A lack of resources made this a challenge to research, but this only increased my determination to find the secrets behind what this shape is truly concealing from us. It is a bit 'waffley', but I tried my best to link this with random concepts I thought of. I tried to make the work relatively formal, so I'm hoping that it worked out decently for my first long math essay.

All graphs / diagrams are either coded in python or they are from Desmos.

Special thanks to Mrs Atkins for proof reading.

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18. The code used to provide these 3d representations derive from myself and some help provided by ChatGPT.