The Mathematics behind Gears and Gear Systems

1 Introduction

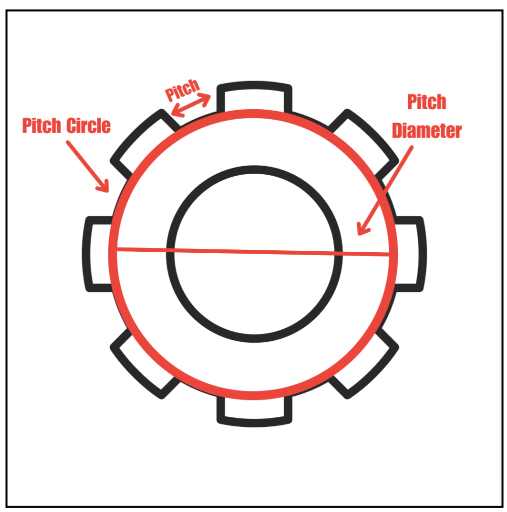
Gears are one of the many revolutionary inventions that have been used by humans since the 3rd century BCE. They trace back to the ancient Greek and Roman civilizations- who, mind you, didn't have the luxuries of using digital smart watches to keep track of time (these are the same people who literally used sundials before inventing mechanical clocks). Fast forward to over two thousand years later, and despite having all this fancy modern advanced technology, we still use gears to keep clocks going and to get to places using our cars. But how exactly do gears even work and what makes them so special? Well, I'll tell you something, it all lies in the mathematical composition of the gear itself and how they function in systems of multiple gears. By understanding these gear aspects, we can then manipulate them to our desired speed, power, rotation, and torque.

2 What is a Gear?

You may be asking yourself, "What even is a gear?" Well, a gear is basically a toothed wheel that is looking for friends, it interlocks with other gears making a system. Gears work by transmitting rotational motion and/or torque to the mechanism. To give you a comparison, gears are kind of like the people who work behind the scenes of movies, they keep the show running but are rarely, if not ever, recognised, just as gears keep the mechanical systems going but aren't the main focus.

One key characteristic of the design of gear is its circular shape. Sure, there are non-circular gears but circular gears happen to be the most versatile and reliable type of gears as they are (non-officially) the best type of gear . This is because circles are by far the coolest shapes ever! Circles are the only shapes that have the same distance at every given point in the circumference.When the gears rotate together in systems, the consistency of the distance between them is crucial because then the gears system will work no matter what position one gear is in, relative to the other. (Yes, I know that gears aren't 2d circles- they are technically 3d cylinders- but we are focusing on the top-down view of the gear which appears to be a circle.)

Another key of gear design is "pitch" (and no, not musical pitch.) In the context of gears, pitch refers to spacing between the teeth on a gear which play a crucial role in gear systems as it determines how tight or loose the gears in a gear system will be. For example, if the pitch of two gears is too large (the teeth are far apart) they probably won't mesh together properly making it so the gears may slip or simply cause unwanted movement amongst the



gears. We also have diametral pitch which is basically how big the tooth is. Diametral pitch is determined by how many teeth are on the gear per inch (weird, right?). When referring to a pitch circle, it is basically an imaginary circle that goes around the circle, representing the average diameter of the gear. This circle determines how the gears will fit together, knowing the pitch circle is crucial if you want your gear systems to function properly because it determines the efficiency, wear rate, and gear ratio of gear systems. Alongside the pitch circle, we also have pitch circle diameter (PCD) because, well, all circles have a diameter. PCD refers to the diameter of the pitch circle (shockingly). We can calculate the PCD of a gear by using this equation:

$$\text{PCD} = \frac{N}{P}$$

Where

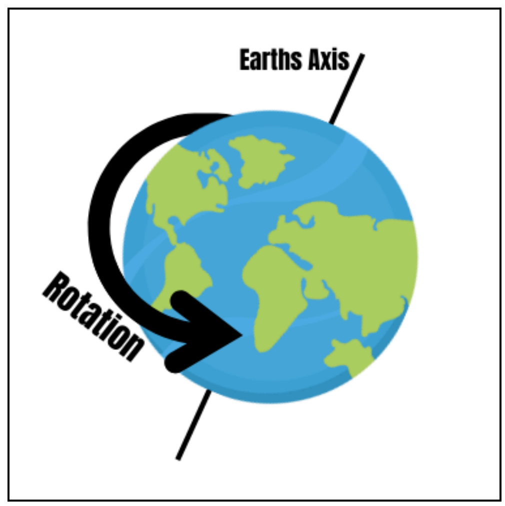
- PCD is pitch circle diameter
- N is the number of teeth on the gear
- P is the diametral pitch

Let's say we have a gear with 24 teeth and a diametral pitch of 3 teeth per inch. This would make the pitch circle diameter be equal to:

$$\text{PCD} = \frac{24}{3} = 8 \text{ inches}$$

This tells us that if we want our gear to mesh properly with other gears, to have a steady and accurate gear system, those gears will have to also have a PCD of 8 inches.

Once the gears start operating, they transmit rotational motion. Rotational motion refers to the circular movement of an object around its own axis. For example, the Earth's roughly twenty-four hour rotation along its axis is a form of rotational motion. In gears, rotational motion is seen when we have them in systems. If you turn one gear, all the other gears will move using the rotational motion of the gear that was turned. The main takeaway from rotational motion is that an object rotates at a fixed axis and all the points on the object move in a circular path in the same direction.

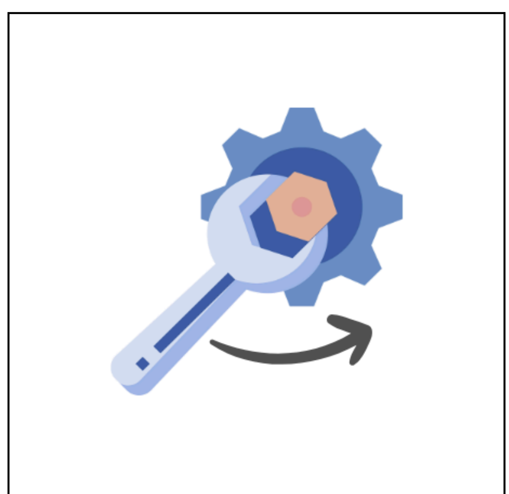


Torque, on the other hand, is the twisting force produced from the rotational motion of the gear(s). Imagine this: You are trying super hard to turn a wrench to tighten a bolt. As you are forcing this wrench to turn the bolt, you finally do. That force went to the bolt, causing it to rotate and tighten. The force that you just applied is an example torque.

To calculate the torque total torque of a typical gear would be something like this:

$$\tau = F \cdot r$$

Where:



- τ represents torque
- F is the amount of force applied
- r is the distance from the axis of rotation to the point where the force is applied. (In the wrench and bolt situation r would be the distance between the end of the wrench and the bolt)

While torque describes the force, power is the big deal when it comes to gears. Power depends on the torque and speed of which the gear is rotating at. We can calculate the power by using this equation:

$$P = \tau \times \omega$$

Where:

- P represents power (in watts)
- τ is torque
- ω is the angular velocity

I know I snuck angular velocity in this equation without describing it beforehand but it's really an easy concept to understand. Essentially, angular velocity in gears is how fast the gear is rotating along its axis or the rate of rotation.

3 Types of Gears

There are many different types of gears, with each type designed for a different purpose- kind of like how each person has a unique set of skills and strengths. Each different gear will possess a similar circular shape but will include some extra unique characteristics. Some of the most common types of gears are spur gears, helical gears, bevel gears, and crown gears.

- Spur gears are the most commonly used gears of all and are usually the gears that come to mind when you think of gears. Spur gears have straight teeth around the circumference of the gear or the wheel, making them the most simple type of gear. To calculate torque, we just use the basic equation given earlier:

$$\tau = F \cdot r$$

- Helical gears, on the other hand, have teeth that are cut at an angle instead of being straight like spur gears. This makes them much stronger than spur gears. Something to consider about helical gears is that to calculate the torque of them you would need to use a slightly different equation from the spur gear due to the angle of the teeth. That equation would look like:

$$\tau = F \cdot r \cdot \cos(a)$$

The reason why we add $\cos(a)$ to the equation is because it factors in the angle of the teeth, a. We have to add this in order to get an accurate calculation of the amount of torque transmitted.

- Bevel gears are sort of described to have a cone-like shape, rather than cylinders. Their teeth can be either straight, like spur gears, or slanted, like helical gears. Bevel gears are generally used when you need to change the direction of rotation between two intersecting shafts. Typically, to calculate the torque of bevel gears, you would use this equation:

$$\tau = F \cdot r \cdot \cos(\beta)$$

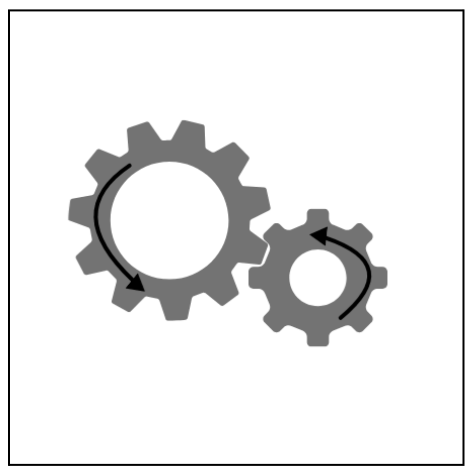
We use cos(β) to express the angle of its cone-like shape. The pitch angle, β, is used to describe how the teeth are oriented along the cone-like shape of the gear.
- Crown gears are a type of bevel gears and are one of the most common types of bevel gears. Their teeth have a 90-degree orientation in which the teeth are perpendicular to the wheel of the gear. For crown gears, the equation is actually pretty simple as it is pretty much the same for bevel gears, with it being:

$$\tau = F \cdot r \cdot \cos(90^\circ)$$

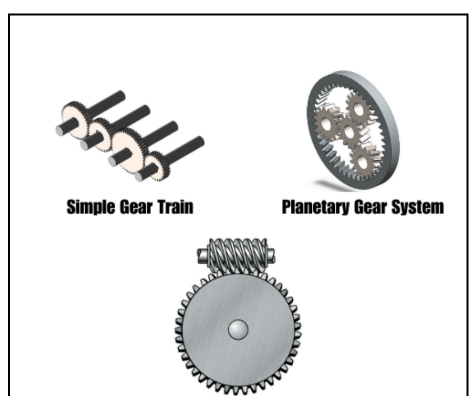
We are able to use $\cos(90^\circ)$ in this equation because we now know the angle in which the teeth are at, 90 degrees, making it so that we are able to just plug it in the equation that we use for bevel gears.

4 Gear Systems

Gear systems are very important for the function of gears; without them, gears would be pretty much useless as they won't be able to transmit any of their motion. Gear systems consist of multiple gears of any size that work together. They work together by interlocking their teeth with one another and transmitting motion and torque to each other. The main gear of a gear system is called the driver gear or the input gear, it starts the movement of the system and transfers rotational motion to all the other gear(s) in the system, the driven gears or output gears. The driven gear will rotate in the opposite direction of the driver gear unless an idler gear is placed in between them. An idler gear is essentially just gears that are there to change the direction of a gear and to connect gears that are too far apart without changing the speed of the gear system.



In a gear system, the type of gears used and the arrangement of them are a very important factor and can vary depending on what you need the gear system to do. In addition, there are many different types of ways to set up gear systems. Some examples of different gear systems are a simple gear train, planetary gear system, and worm gear systems.



- Simple gear trains are by far the most common type of gear system since they consist of spur gears that are simply placed directly next to one another. When powering mechanisms, simple gear trains typically have 2-3 gears but they can have more. However, it's quite impractical to have more gears unless you need them due to your main gears being too far apart, essentially putting gears to fill up the gap.
- Planetary gear systems or epicyclic gear trains, are gear systems that resemble solar systems and are mostly made up of spur gears. Planetary gear systems have a sun gear (the gear in the middle), planet gears (the gears around the sun gear), and a ring gear or annular gear (the gear around the planet gears). To keep the gears in this formation, a planet carrier is used since it not only keeps the gears together but also allows movement. Planetary gear systems have at least three gears in total(with one sun gear, one or more planet gears, and one ring gear) alongside the plane carrier.
- Worm gear systems consist of a worm gear and a worm wheel. The worm gear is a screw type gear that has a helical thread wrapping around the body of the gear (I know this explanation isn't very detailed so I provided a photo that includes a worm gear above). Worm wheels are a type of gear similar to the spur gear but have teeth angled in a certain way to mesh perfectly with the worm gear. This gear system comes particularly in handy because of the fact it prevents backdriving (the worm gear can turn the worm wheel, but the worm wheel can't turn the worm gear) and takes up little space.

5 Gear Ratios

Gear ratios have a very crucial role when it comes to gear systems. Essentially, a gear ratio is the relationship of the amount of rotations between the driver gear and driven. Gear ratio factors in the amount of teeth each gear has.
We are able to calculate the gear ratio (GR) between two gears by using an equation like this:

$$GR = \frac{Number\ of\ Teeth\ on\ Driven\ Gear}{Number\ of\ Teeth\ on\ Driver\ Gear}$$

The gear ratio is the key to determining the speed and torque of a system. For example, if our driven gear has 12 teeth and our driver gear has 48 teeth, the gear ratio is:

$$GR = \frac{12}{48} = 1:4 = \frac{1}{4}$$

This means that the driver gear can complete 4 rotations when the driven gear has only completed 1, resulting in more speed but less torque. System.
Now, if we had 48 teeth on our driver gear and 12 on the driven gear, the gear ratio would be:

$$GR = \frac{48}{12} = 4:1 = 4$$

This means that the gear system would now have more torque but less speed. This is because with a high gear ratio, the driven gear makes fewer rotations than the driver gear. This then results in the system having less speed but more torque. The increase in torque is there because the driver gear now has to put more force into turning the driven gear.

If we are looking to calculate the total gear ratio of a system (TGR), we would need to add on to the equation depending on what type of gear system we are trying to solve for. Now, since simple gear trains are the most common type of system, I will only be providing a TGR calculation for them. For a simple gear train, we would simply just calculate the gear ratio for every two gears in the system and then multiply them together. For example, let's say we have a simple gear train of three gears, Gear 1 (the driver gear), Gear 2 (the idler gear), and Gear 3 (the driven gear). The number of teeth on Gear 1 is 10, on Gear 2 there are 8, and on Gear 3 there are 12 teeth. What we first need to do is find the gear ratio of Gear 1 and Gear 2. We can do this by plugging in the teeth in the equation given previously so we should have an equation that looks like this:

$$GR = \frac{Number\ of\ Teeth\ on\ Gear\ 2}{Number\ of\ Teeth\ on\ Gear\ 1} = \frac{8}{10} = 4{:}\,5$$

We then need to find the gear ratio of Gear 2 and Gear 3:

$$GR = \frac{Number\ of\ Teeth\ on\ Gear\ 3}{Number\ of\ Teeth\ on\ Gear\ 2} = \frac{12}{8} = 3{:}\,2$$

Now, in order to calculate the total gear ratio (TGR)we would need to multiply them together:

$$TGR = \frac{4}{5} \times \frac{3}{2} = 12{:}\,10 = 6{:}\,5 = \frac{6}{5}$$

So, now we know that the total gear ratio (TGR) of the gear system is 6:5 or 1.2. You may notice that if we didn't simplify the gear ratios beforehand, the idler gear actually would cancel itself out. This is because- as stated early on in this essay- the idler gear only transfers motion to the driven gear and has no interaction besides that.

6 Real World Applications of Gears

You may not realize this, but gears are used in many different types of mechanical systems all around us. Some places where we see gears are in transportation, industrial machines, and clocks. In transportation, gears are practically the main component to making the automobile, bike, or helicopter move. In these types of gear-powered mechanisms, gears play the crucial role of controlling the speed and direction. In industrial machines like conveyor belts or cranes, gears are used for their torque and power. In clocks, gears are used to make the hands move with precision. In addition, gears are also used in elevators, washing machines, wind-up toys, etc. Gears are truly the true heroes of our everyday life as they are seen everywhere whether you are getting to work or washing your clothes, making our lives ultimately easier.

7 Conclusion

Although we've clearly come a long way from the ancient civilizations that gave us the fundamentals of engineering, one thing that will never be replaced are gears. Gears are truly one of the most revolutionary inventions known to humankind. From their mathematically amazing composition and power to their variety uses in everyday life- gears have always been the foundation for innovation and progress.