The Mathematics of Our Brains

Introduction

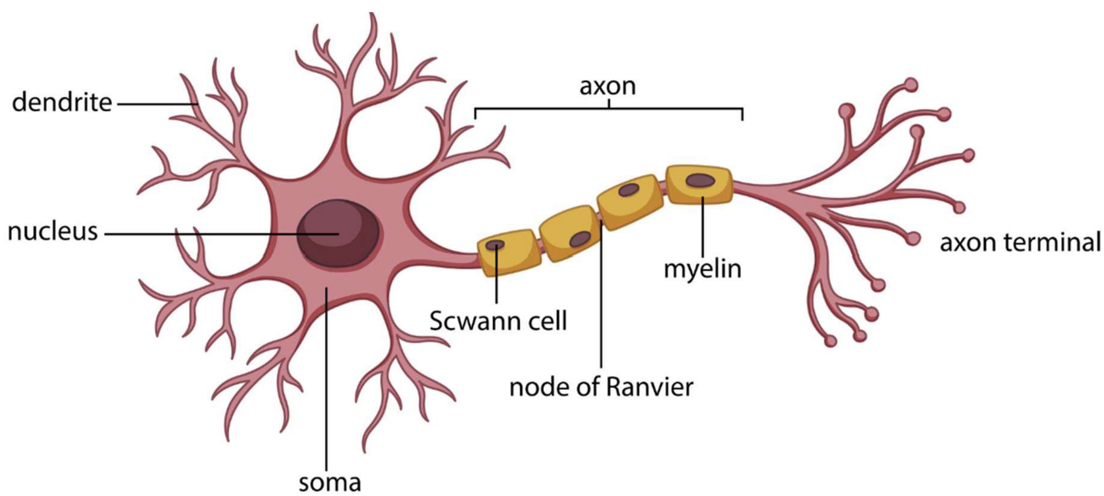
Mathematics has many uses, from engineering to economics, but a rarely talked about subject is mathematical modelling.

What does this mean? Mathematical modelling is when we use mathematical equations to describe a real-world problem to better understand it. For example, modelling can be used in everything from robotics to astrophysics, but what most interests me is the mathematical modelling used to create representations of our brains, so that is what I will delve into in this essay.

Background Knowledge

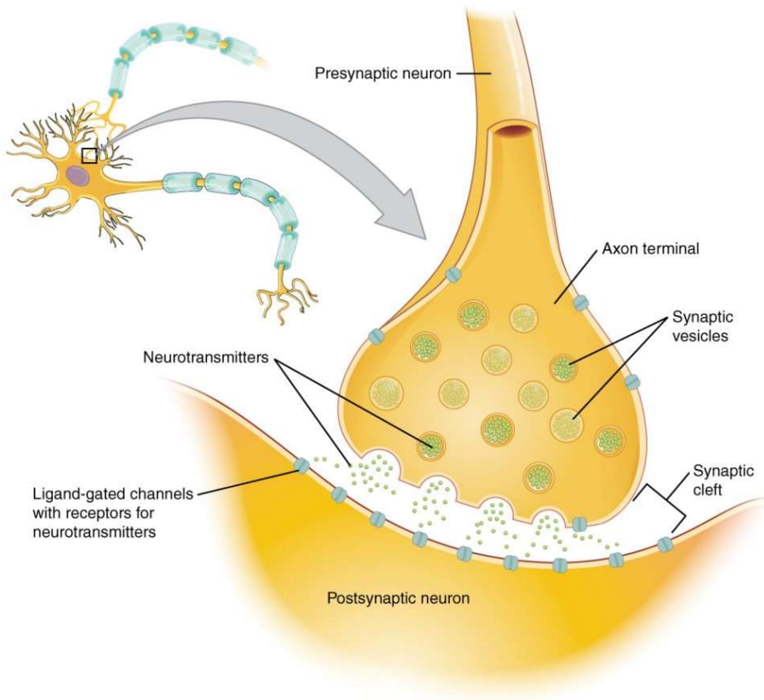
To understand how to model the brain, we must first understand the biology of it.

Our brain consists of cells called neurons. The structure of a neuron is shown below:

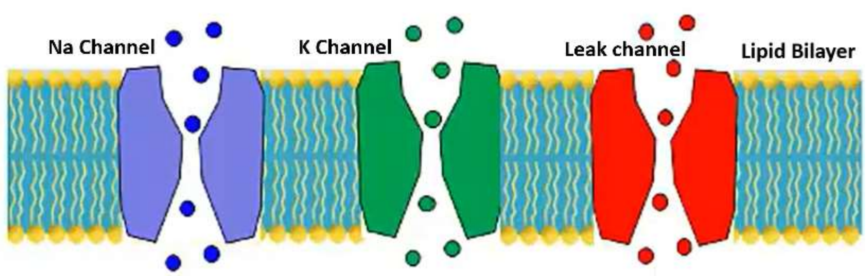


In this essay, we'll be considering how to mathematically model the communications between neurons. Biologically, neurons communicate through chemical neurotransmitters releasing into synaptic cleft (the space between 2 neurons) from one neuron to the next. This can cause an electrical signal in the postsynaptic neuron (the second neuron), which in turn can cause that neuron's neurotransmitters to be released and so on.

In short, when neurons communicate with each other, they fire electrical signals to other neurons. These electrical signals are called action potentials or spikes.



Within the synapse, there are channels. On the diagram above these are shown as the Ligand-gated channels.



Hodgkin and Huxley discovered these channels in 1952 while dissecting a squid's brain, specifically the axons of its neurons. We'll learn more about their neuronal model later in this essay.

Remember that, in reality, there isn't a specific leak channel: this is just a visual representation of how some ions can diffuse through the lipid bilayer and be lost.

The LIF Model

The first model I'd like to discuss is the LIF Model (The Leaky Integrate-and-Fire Model).

This model was investigated by Louis Lapicque in 1907. Firstly, he proposed a 'perfect' integrate-and-fire model without the leakage. The equation for this is shown below:

$$I(t) = C\frac{dV(t)}{dt}$$

This equation is just a time derivative of the law of capacitance, which is C=Q/V, where C is the capacitance, Q is the charge and V is the voltage. This means that capacitance is just the ratio between the charge and the voltage, or in other words, it is the ability of an object to store electric charge.

We can rearrange the law of capacitance to be Q = CV.

Lapicque's equation is a time derivative of that law, which means that he's done the following process:

$$Q = CV$$

$$\frac{dQ}{dt} = C\frac{dV(t)}{dt}$$

$$I(t) = C\frac{dV(t)}{dt}$$

Remember: $I\ (current) = \frac{Q\ (charge)}{t\ (time)}$

The important thing to recognise is that this model is only applicable until a certain voltage. Because after this threshold is met, an action potential is said to have occurred and the membrane potential resets itself. This is modelled with the following equation.

$$if\ V(t) \geq V_{th} \qquad then\ V(t) = V_{reset}$$

Next, Lapicque proposed the leaky integrate-and-fire model which, unlike its 'perfect' counterpart, involves a leak term. The other model is dubbed 'perfect' because it depicts what would occur in a perfect world – if no ions would diffuse through the membrane and be lost. However, this is not how our brains work in reality. Lapicque's leak term makes the model more accurate.

The equations to get to the LIF model are shown below:

$$I(t) = C\frac{dV(t)}{dt}$$

$$C\frac{dV(t)}{dt} = I(t)$$

$$C\frac{dV(t)}{dt} = I(t) - \frac{\boldsymbol{V_m(t)}}{\boldsymbol{R_m}}$$

The added leak term is shown in red. V_m is the voltage across the cell membrane and R_m is the membrane resistance. Similarly to the previous equation, this model is also only valid for until the threshold V_{th} is reached and the membrane potential is reset.

Remember: $I\ (current) = \frac{V\ (voltage)}{R\ (resistance)}$

The leak term refers to the current lost due to ions diffusing through the lipid bilayer we saw earlier in the 'Background Knowledge' section.

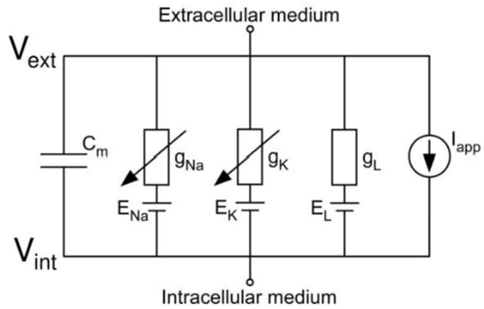
Now we've learnt about the LIF model – a mathematical representation of the current through a neuron.

Hodgkin-Huxley Model

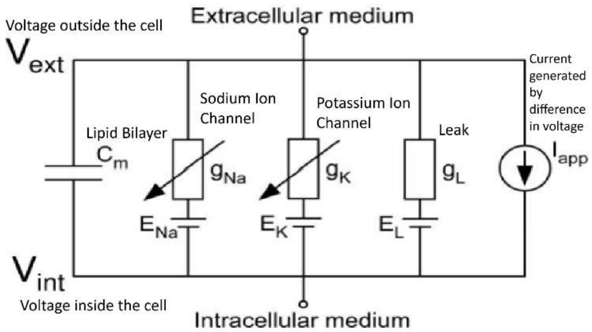
As mentioned earlier, another model I'd like to talk about is the Hodgkin-Huxley Model. Alan Hodgkin and Andrew Huxley received the Nobel prize for the creation of this model in 1963, a well-earned award due to how their model has shaped our understanding of how neurons communicate within our brains.

The Hodgkin-Huxley Model is a mathematical framework for how action potentials are initiated and propagated in neurons. This means it shows how spikes form by calculating the current within a neuron using the voltage. It does this by modelling each component of a neuron as an electrical element.

Below we have Hodgkin-Huxley's representation of a neuron using a circuit diagram.



Below, I'll annotate the diagram to show what is happening at each stage.



The Hodgkin-Huxley model is an example of a biophysical model, which means that its closer to being biologically accurate than other models, like the LIF model. This also means it tends to be more complex.

The difference between the voltage inside the cell and outside the cell is the ion gradient as it is caused by the difference in ions inside and outside the cell. This creates the potential difference that then generates the current that flows through the neuron.

The Na refers to the sodium channel and the K refers to the potassium channel. They are represented by variable resistors as the amount of ions that pass through the channels can differ based on the electrical signals that reach the synapse. The g_L resistor refers to the ions 'leaking' through the lipid bilayer – this resistor is not variable because how many ions diffuse out is not controlled.

So to summarise:
Resistor – ion channels
Capacitor – lipid bilayer
Battery – ion gradient

Hodgkin and Huxley also created the formula below to demonstrate their model mathematically.

$$I = C_m \frac{\mathrm{d}V_m}{\mathrm{d}t} + g_K(V_m - V_K) + g_{Na}(V_m - V_{Na}) + g_l(V_m - V_l)$$

This formula may seem daunting at first, but let's break it down to understand it.

$$\boldsymbol{I = C_m \frac{dV_m}{dt}} + g_K(V_m - V_k) + g_{Na}(V_m - V_{Na}) + g_l(V_m - V_l)$$

The first part of the formula derives from Lapicque's 'perfect' integrate-and-fire model, which we've already seen.

$$I = C_m \frac{dV_m}{dt} + \boldsymbol{g_K(V_m - V_k)} + g_{Na}(V_m - V_{Na}) + g_l(V_m - V_l)$$

g_k is the conductance. The conductance is the measure of how easily electrical current can pass through. This conductance is both voltage and time dependent, meaning it can change based on the potential difference across the membrane and the time since the previous spike.
V_m is the voltage across the membrane (the total voltage).
V_k is the voltage across the potassium channel.

When put together, this subtracts the current across the potassium channel from the overall equation.

$$I = C_m \frac{dV_m}{dt} + g_K(V_m - V_k) + \boldsymbol{g_{Na}(V_m - V_{Na})} + g_l(V_m - V_l)$$

The next term is very similar to the previous term except instead of the potassium channel, it subtracts the current across the sodium channel.

$$I = C_m \frac{dV_m}{dt} + g_K(V_m - V_k) + g_{Na}(V_m - V_{Na}) + \boldsymbol{g_l(V_m - V_l)}$$

The last term is the same as the previous two terms but it subtracts current from the ions that have diffused through the lipid bilayer and been lost.

Therefore, Hodgkin and Huxley's formula depicts the total current passing through the membrane of a neuron cell with sodium and potassium channels.

Conclusion

To conclude, within this essay, we've explored two mathematical models commonly used in computational neuroscience: the LIF model and the Hodgkin-Huxley model, both of which represent the current passing through a neuron.

In a more general sense, these models help to explain the mechanisms behind the nervous system and bring us a step closer to understanding how our brains function. When implemented on a larger scale, these models can be used to restore brain function by mimicking neurons through prosthetics. This technology can potentially lead to advancements in curing conditions such as dementia, Parkinson's and paralysis.

Obviously, within neuroscience, there are far more mathematical models that I could talk about, but I've condensed this essay to the two that most interested me for their uses and the elegance of the maths involved.

I hope that, through this essay, you've leant more about the interesting world of mathematical modelling and how it can be used to understand more about our own biology.

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