The Brachistochrone(-osaur)

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1 Introduction

A roar echoes through the night. The ground trembles with each stride as you feel the incoming danger. The hair on the end of your arms stands up. You prepare yourself.

"RAAAAAAH!"

You see it, pulsating with rage and roaring with pride. You thought you were ready, but your feet stay frozen, too scared to move. There before your eyes is the Brachistochrone. No, not the Brachiosaurus; in fact, it's not remotely related to anything you might expect.

In the late mathematical feuds of the 17th century, when the world of science was dominated by the five great mathematicians – Isaac Newton, the Bernoulli Brothers, Guillaume de l'Hôpital and Gottfried Leibniz – in an effort to declare himself, Johann Bernoulli, the finest intellect of his era, placed a challenge before the scientific community that came to be known as the Brachistochrone problem. The problem was as follows:

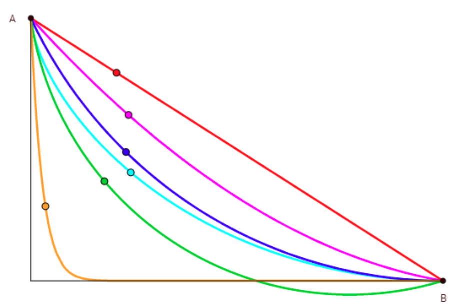


Figure 1: Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time

The purpose of posing this problem was thought to be a ploy by Bernoulli and Leibniz to provoke Isaac Newton over the Great Calculus War of the 1680s, mainly between Leibniz and Newton over the discovery of differentiation and integration. Ultimately, six months after posting this problem, Johann, in his pride, assumed he had confuddled the great Isaac Newton, who had submitted his solution anonymously. Upon encountering it, Johann Bernoulli is said to have famously remarked, 'I know the mark of a lion by his claw,' as he marvelled at the answer Newton had provided.

But, putting my patriotism aside, this isn't the tale of how Isaac Newton asserted his rule over the mathematicians of the 17th century but rather about the nature of the Brachistochrone itself as well as appreciating and acknowledging the elegant, unique and beautiful work of Johann or Jacob Bernoulli – we don't know for sure, as it is said that Johann, in an attempt to supersede his brother, passed off Jacob's solution as his own.

1.1 A little more about the problem

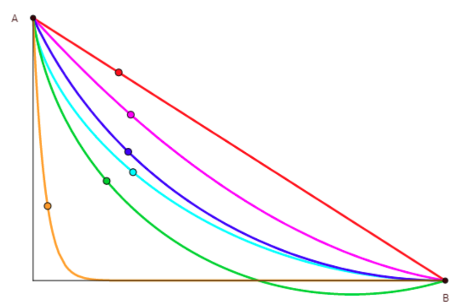


Figure 2: Path of quickest time

Let's take a closer look at the essence of the problem: given two points in a vertical plane, with the second point lower and not directly beneath the first, what is the path that the particle will take to reach the second point in the least amount of time whilst starting from rest and moving under gravity alone?

Intuitively, one would assume that the quickest path would be the path of shortest distance, a straight line connecting A and B, or perhaps a simple arc such as that of a circle. In fact, Galileo Galilei had posited this solution almost half a century prior to Johann Bernoulli publicising this problem. Though Galilei's circular path allowed for faster acceleration at the start, it wasn't actually the optimal path for minimising the total travel time. The problem was that the particle does not gain enough speed early on, and the later, flatter paths of the circular arc slow it down too much. Regardless, it is evident that the core of the problem revolves around the idea that objects accelerate as they fall, so a steeper initial path might allow the particle to pick up speed early, even if the total distance is longer, leading to a naturally curved shape.

2 Approaching the speed of light

Before we get into Bernoulli's solution, it is necessary to understand the principles of light and optics that his proof is nested in. Through an intuitive understanding of Snell's law. This is a result in physics which describes how light bends when it travels from one medium into another where its speed changes. Consider when a beam of light travels from air to water. The speed of light is a little slower in the water than in the air. Think of it as running from the beach to the ocean; you slow down as you enter the sea. Similarly, as light enters the water, it ever so slightly bends towards the normal.

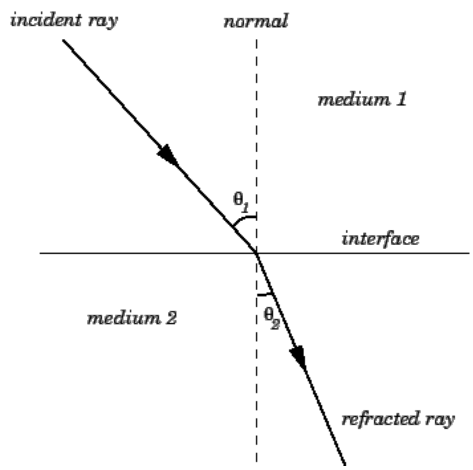


Figure 3: Beam of light as it enters a denser medium

2.0.1 Why?

An intuitive way to understand this phenomenon is through Fermat's principle of optics, which states that a beam of light travelling from A to B does so along the fastest path possible, as if nature itself seeks out the path of minimum time. Through this we can employ a clever analogy to find this said path, without diving too deeply into calculus (don't worry, we'll keep it light), which converts this from a problem of finding the minimum time to a problem of minimum potential energy

In our case, imagine placing a rod on the border between the two mediums and then placing a ring on the rod which can pivot along it. Imagine connecting a spring from point A to the ring and another spring from point B to the ring. The layout of this system represents a possible path that the beam of light can take. To optimise this setup so that the potential energy in the springs is analogous to the amount of time it would take for light to travel that along the said path, we have to assume that the spring has constant tension that is inversely proportional to the speed of light in that specific medium. For ease, let us assume medium one is air and medium two is water.

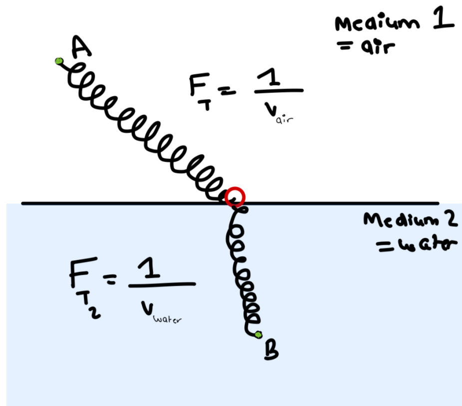


Figure 4: Snell's Law analogy with springs

It is worth noting that these springs are unphysical; constant tension springs cannot exist in nature. However, the principle of the system wanting to minimise its total potential energy will still hold and will be the basis of our proof. Now with it set up, we can find the minimum state by simply resolving the forces the springs make with one another in the horizontal direction. As the system

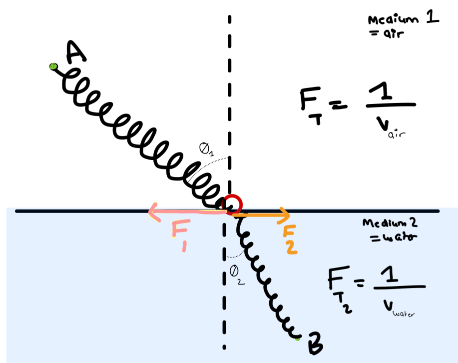


Figure 5: State of equilibrium

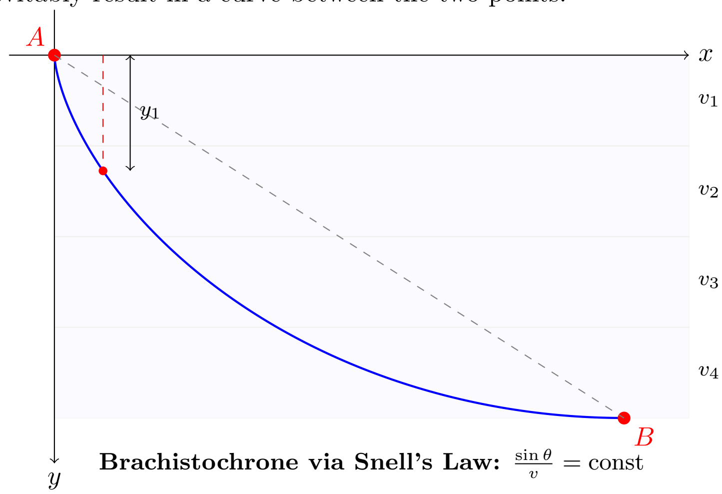
at this point is in a state of equilibrium, the forces must therefore be equal to each other.

$$F_{T_1} sin(\theta_1) = F_{T_2} sin(\theta_2) \tag{1}$$

Using the laws of proportionality we previously assumed $F_t = 1/v_{medium}$, we rewrite the the given equation to yield,

$$\frac{sin(\theta_1)}{v_{air}} = \frac{sin(\theta_2)}{v_{water}} \tag{2}$$

And so we get Snell's law. That says that the sine of theta divided by the speed of light stays constant when light travels between mediums. Where theta is the angle that the beam of light makes between the normal and the boundary between the two media. In the Brachistochrone problem we can imagine that the speed of light increases with depth due to gravity as if it were passing through infinitely thin layers of varying medium, and therefore the path would inevitably result in a curve between the two points.



Brachistochrone via Snell's Law: $\frac{\sin\theta}{v} = \text{const}$

Using the conservation of energy, we know the loss in potential energy of a particle travelling down this path is equal to the particle's total kinetic energy at a given point.

$$mgy = \frac{1}{2}mv^2 \tag{3}$$

Rearranging equation (3), we get,

$$v = \sqrt{2gy} \tag{4}$$

To simplify matters we can say that $\sqrt{2g}$ is equal to k and therefore

$$v_{medium} = k\sqrt{y} \tag{5}$$

So if light is always instantaneously obeying Snell's Law, as it travels from one medium to another, and we have deduced that the path when it travels through infinite medium is a some curve between A and B, we can say that

$$\frac{sin(\theta)}{k\sqrt{y}} = C \tag{6}$$

Which can be simplified to,

$$\frac{sin(\theta)}{\sqrt{y}} = Consant \tag{7}$$

Where y is the vertical distance between the point on the curve and the start of the curve, and the constant is independent of the point on the curve, as we have deduced from Snell's law. At this point, the equation we have derived may not mean much, but pretty soon, with some neat piece of geometry, we will finally find the path of quickest descent.

3 A cy-cy-cyloid???

Imagine you are walking alongside a giant wheel, such as a bike tire, that is gently rolling across a smooth horizontal path. Before it starts moving, you take a glowing marker and make a tiny dot right on the edge of the tyre. Now as the wheel begins to roll, the glowing dot leaves behind a sparking trail in the air as it revolves with the wheel. If your imagination is strong enough, you may notice that the dot isn't simply going forward; it dips down towards the ground, then swoops up into the air, over and over, making perfect symmetrical humps, like a series of arches laid end to end. This shape is the cycloid, a curve traced by a point on the edge of a wheel as it rolls smoothly along a flat surface

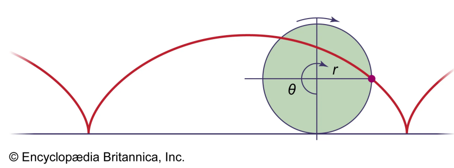


Figure 6: A cycloid

In our case, let's do the same thing, but instead of a wheel rolling on flat ground, imagine it rolling along the ceiling. Visualise a point on the rim as done previously. The point where the wheel touches the ceiling, C acts an instantaneous centre of rotation, just like a pivot. The line segment PC is therefore perpendicular to the tangent of the cycloid path at point P. The tangent intersects the base of the circle to form a vertical. Let the angle between the tangent and the vertical be θ . By geometric similarity the angle between the segment PC and the horizontal is also equal to θ .

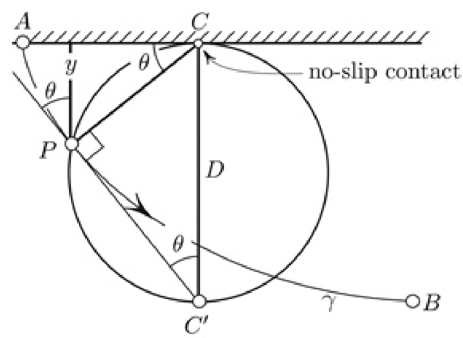


Figure 7: An upside down cycloid

From this diagram we can see that PC is $Dsin(\theta)$, and this length times the sine of theta again gives us the vertical height between P and the start of the curve:

$$y = Dsin^2(\theta) \tag{8}$$

Rearranging this equation gives:

$$\frac{1}{\sqrt{D}} = \frac{sin(\theta)}{\sqrt{y}} \tag{9}$$

Since the diameter of the circle remains constant as the wheel rotates, this implies that the

$$Constant = \frac{sin(\theta)}{\sqrt{y}} \tag{10}$$

which is exactly the Snell's Law property we derived earlier!!

So this tells us that the path of quickest descent between two points, A and B, on a vertical plane is that of the path traced by the cycloid, rotating along the horizontal plane. Among all possible curves, only the cycloid maintains this ratio $\frac{sin(\theta)}{\sqrt{y}}$ as a constant throughout its length. This is a well-known geometric characteristic of the cycloid. Just like how a beam of light bends as it enters a denser medium where its velocity decreases, the particle on the brachistochrone starts steeply, quickly gains speed, then glides more gently as it approaches the endpoint. The cycloid's shape mirrors this continuously changing 'refractive' behaviour through gravity.

The condition $\frac{sin(\theta)}{\sqrt{y}} = Constant$ leads directly to the differential equation of a cycloid. Solving that equation using calculus of variations and the set of Euler-Lagrange equations (beyond my understanding of calculus) confirms the cycloid as the unique solution.

4 Conclusion

The brachistochrone problem marked a turning point in the history of mathematics, ultimately leading to the development of the calculus of variations by Euler and Lagrange some fifty years later. Its solution, the cycloid, has an interesting characteristic where the time taken on the descent is constant regardless of its starting point. In an ideal scenario, if a particle were to be dropped on the brachistochrone ten centimetres away from the end whilst another were to be dropped fifty centimetres from it, the two particles would reach the end at the exact same time. This duality highlights the cycloid's remarkable properties and its deep connection to natural laws.

I was initially drawn to this problem not only because of its unique and bizarre name (it did play a big role) but also because of how it incorporated such vastly different aspects of science: from optics to energy conservation to circle geometry. I feel it speaks as a testament to the far-reaching grasp of mathematics, as if nature is solving a 'find the minimum' style of question.

The brachistochrone has many applications in the engineering industry, such as using cycloidal curves to minimise impact and mechanical stress in certain gear and cam designs, leading to smoother transitions and increased durability. While the brachistochrone assumes there is no friction (the bane of all elegant solutions), real-life applications often involve resistance. In such cases, engineers turn to iterative processes, such as the Euler method, to find 'near-brachistochrones' suited to real-world scenarios, where the tug of war between velocity and time is interrupted by resistance...

I hope you enjoyed this essay and thank you for reading!!

5 References

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