The Golden Ratio of the Collatz Conjecture

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1 Intro

The Collatz Conjecture originally proposed by Lothar Collatz in 1937 displays a simple problem to understand, yet holds a complex underlayer which creates difficulty in proving the conjecture. It states that for any positive integer n, if it is odd, multiply it by 3 and add 1, but if even, divide by 2. Eventually, this will always reach a repeating sequence of 4, 2, 1. This can also be written in modular arithmetic notation as:

$$C(x) = \{_{x/2x=0(MOD2)}^{3x+1x=1(MOD2)} \tag{1}$$

Whilst this problem is often given to children as an exiting and cool task to play around with, or thought of by some as a waste of time quoting "Mathematics may not be ready for such problems." - Paul Erdos, for me it is the many patterns and secrets that provide a gateway into the exploration of the vast world of mathematics.

In this I will explore how the fibonacci sequence interacts with the Collatz conjecture and prove the proportions of numbers that are odd, and that are even after applying the series k times.

2

The main problem that has stumped proofs is when applying the sequence to n integers, there is no apparent correlation to the starting number and the number of iterations required to reach 1 and enter the repeating sequence. You would think that as the number increased so would the number of steps yet this is not the trend. There are many 'proofs' which currently exist but are not accepted as a total and complete proof, for example, it has been shown that all numbers tend towards 1, and a large scale project by BOINC that has tested all number up until 2^{63}. Surely this is solid proof?

n	Iterations	n	Iterations
1	0	70	14
2	1	71	102
3	7	72	22
4	2	73	115
5	5	74	22
6	8	75	14
7	16	76	22
8	3	77	22
9	19	78	35
10	6	79	35

Figure 1: Some starting numbers and their correlating iterations to reach 1

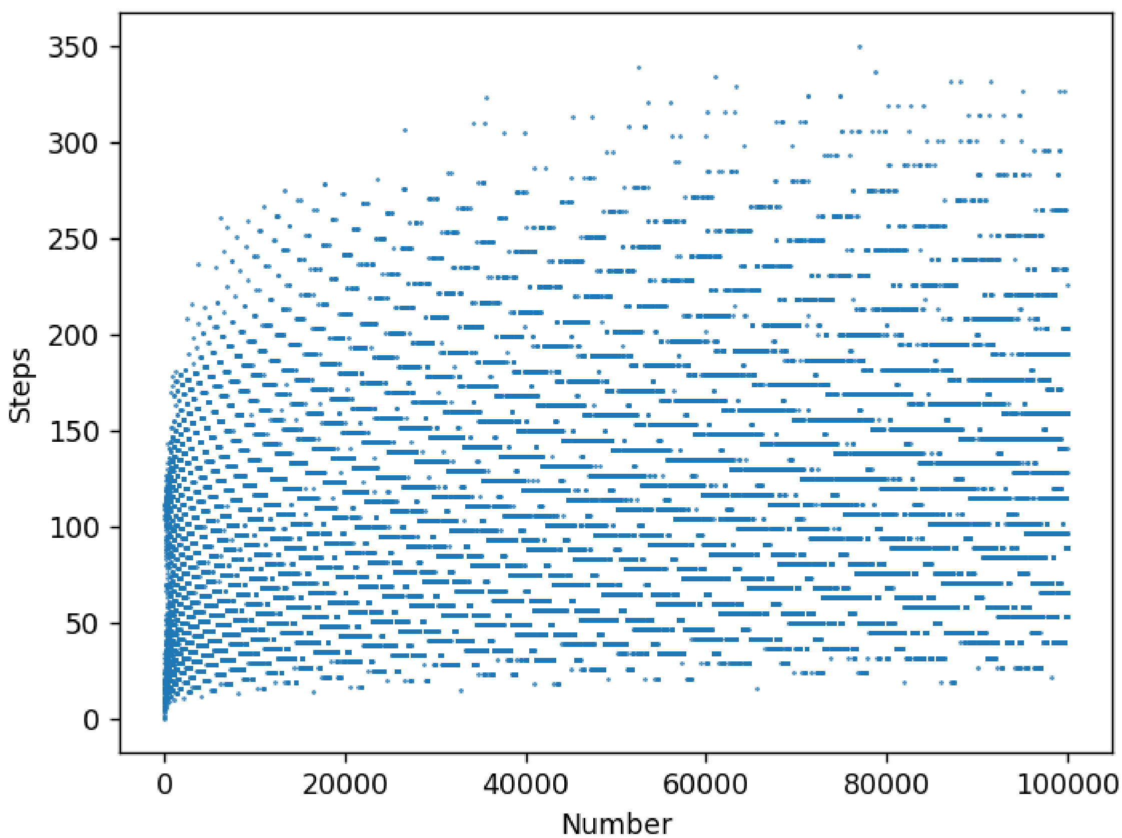
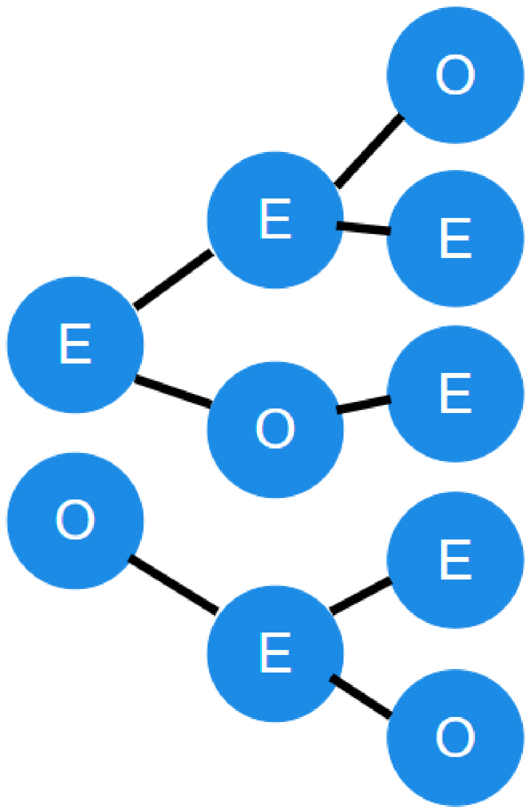


Figure 2: Start Number vs number of steps to reach 1
When we apply the rules of the conjecture to a large set of integers, a wave-like pattern begins to form giving it the alternative name; the hailstone sequence.

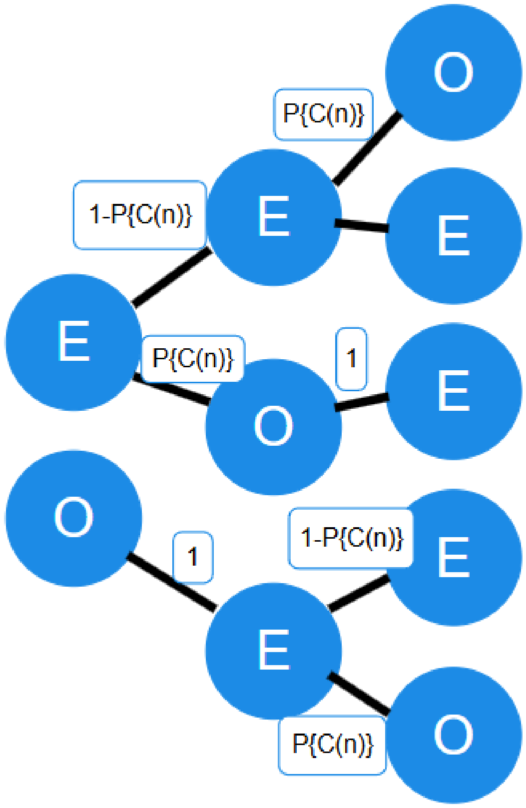
However, if we look at each individual number and the values on each of their iterations, there is a more subtle pattern. For any even number, dividing by 2 can result in either O(odd) or E(even). An odd number always results in even:

$$f(n) = 3n + 1$$
$$f(2n+1) = 3(2n+1) + 1 = 6n + 4 = 3(2n) + 4 \tag{2}$$

This can then be written as a tree diagram displaying how the type of number changes as the iterations increase.



When observing specifically the probability of odd values, starting that any chosen number has equal probability of being odd or even, p=$\frac{1}{2}$. Letting the outcome of a single iteration being denoted as C(n), the probability can be written as P{C(n)}. This transforms the tree diagram into:



$$P\{C^0(n)\} = \frac{1}{2}$$

$$P\{C^1(n)\{= \frac{1}{2} - \frac{1}{2}(P\{C(n)\}) \tag{3}$$

By manipulating this, a general formula can be created

$$p\{C^k(n)\} = \frac{1}{2} - \frac{1}{2}(p\{C^{k-1}(n)\}) \tag{4}$$

If we then treat the expression recursively, terms begin to to cancel out which leaves us with

$$p\{C^k(n)\} = \frac{1}{2} - \frac{1}{2}(\frac{1}{2} - \frac{1}{2}(p\{C^{k-2}(n)\}))$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + (-\frac{1}{2})^k \tag{5}$$

This can be treated geometrically using a = $\frac{1}{2}$ and r = -$\frac{1}{2}$ as lim k$\rightarrow\infty$,

$$
\begin{aligned}
p\{C^{\infty}(n)\} &= \frac{a}{1-r} \\
&= \frac{\frac{1}{2}}{1-(-\frac{1}{2})} \\
&= \frac{\frac{1}{2}}{\frac{3}{2}} \\
&= \frac{1}{3}
\end{aligned} \tag{6}
$$

This formula can also be expressed as a summation series

$$\sum_{r=1}^{\infty} -\frac{1}{(-2)^r} = \frac{1}{3} \tag{7}$$

To prove this, I tested 20000 randomly generated numbers and iterated the conjecture between 1 and 200 times and the results supported the equations.

3

The Fibonacci sequence created by the Italian mathematician Leonardo Pisano Bigollo to solve the rabbit breeding problem proposed in *Liber Abaci.*

In 1843, *Binet's Closed Fibonacci Formula* was discovered by J. Binet.

$$
\begin{aligned}
F(n) &= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n) \\
&= \frac{1}{\sqrt{5}}(\Phi^n - (\frac{1}{\Phi})^n
\end{aligned} \tag{8}
$$

Before I dive into how these 2 topics link, the question that must be answered is "What is Phi?"

The letter Phi ϕ originates from the Greek alphabet but is used to denote the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618033988749.... \tag{9}$$

Take a strip of paper and cut it into 2 different sized pieces. The golden ratio is where the ratio between the length of the longer piece and the entire original piece is equal to the ratio of the length of the smaller piece and the bigger piece.

Iteration	Total	Odd	Even
0	2	1	1
1	3	1	2
2	5	2	3
3	8	3	5
4	13	5	8
5	21	8	13
6	44	13	21

Figure 3: Returning to the tree diagram from earlier, it can be seen that the number of available branches on each iteration follows this sequence starting from 2, 3, 5, 8 etc. Additionally, in each iteration, the number of branches with O and number with E also follow the sequence.

Using this knowledge it is possible to know from the number of iterations of the conjecture, n, it is possible to determine the number of pathways available and the proportion of O pathways. Observing the table above, the sequence around the O values are 2 terms behind the total, allowing for a formula to be written.

$$P\{OddPathway\} = \frac{\frac{1}{\sqrt{5}}(\Phi^n - (\frac{1}{\Phi})^n)}{\frac{1}{\sqrt{5}}(\Phi^{n+2} - (\frac{1}{\Phi})^{n+2})}$$

$$= \frac{\Phi^n - (1-\Phi)^n}{\Phi^{n+2} - (1-\Phi)^{n+2}} = \frac{\Phi^n - (1-\Phi)^n}{\Phi^2\Phi^n - (1 - 2\Phi + \Phi^2)(1-\Phi)^n}$$

$$\frac{1}{P\{OddPathway\}} = \frac{\Phi^2(\Phi^n - (1-\Phi)^n)}{\Phi^n - (1-\Phi)^n} - \frac{(1-\Phi)^n}{\Phi^n - (1-\Phi)^n} + \frac{2\Phi(1-\Phi)^n}{\Phi^n - (1-\Phi)^n}$$

$$\lim_{n ß \infty} \Phi^2 - \frac{(1-\Phi)^n}{\Phi^n - (1-\Phi)^n} + \frac{2\Phi(1-\Phi)^n}{\Phi^n - (1-\Phi)^n}$$
$$= \Phi^2 - \frac{0}{\infty} + \frac{0}{\infty}$$
$$= \Phi^2$$

$$P\{OddPathway\} = \frac{1}{\Phi^2} \tag{10}$$

This shows that the final probability of entering an odd pathway is $1/\Phi^2$.

At the current time of writing this, and hours of groping over equations, trees, Os and Es, I have not managed to create an equation which links these two probabilities together and intertwine a seemingly distant sequence into the expanse of the Collatz conjecture.

4

Imagine you had a computer that could do infinite calculations in very little time. Using that computer you run the Collatz Conjecture through every single integer imaginable an infinite number of times, a third of the resulting numbers will be odd. And will the equations ever be found that link together these incredible mathematical patterns, we might never know, maybe I already have. Or will you figure it out?