IN THE PURSUIT OF DIFFERENTIATION:

An exploration of 17th-century mathematical methods.

Elliott Bathe
April 2025

INTRODUCTION:

Every great field has rivalries. Football has the Manchester Derby; rugby has England vs. France; superheroes have Batman and Joker. 17th-century mathematics was no different: Two outstanding French mathematicians, René Descartes and Pierre de Fermat, are widely considered the biggest rivals in the field of mathematics. From their feuds came many major discoveries, many (creative) insults, and a burning desire to constantly prove each other wrong.

The first of these feuds was the "tangent line problem" where Descartes used a method involving circles (his circle method) and Fermat used a method involving triangles (his adequality method). The "tangent line problem", like many problems in pure maths, had little to no everyday use in the 17th century, yet today the methods associated with this problem are some of the building blocks of modern-day society.

Who should be credited with solving this problem? Whose solution can be regarded as the most elegant?

The "tangent line problem", as stated in its name, is the idea of being able to construct a generalized equation to find the equation of a tangent (a line that only touches and does not intersect the curve) at any given point on a curve.

RENÉ DESCARTES:

René Descartes was a philosopher and mathematician best known for his work on analytic geometry and his quote "Cogito, ergo sum"[1]. Yet one of his lesser-known talents was crafting brutal insults and being purely and entirely petty. A great example of this comes from his work on tangent lines. But for us to explore his pettiness we must first understand his method of finding the tangent line. Descartes first displayed his method in book two of "La Geometrie" (1637).

Descartes started his method by drawing a normal to the curve at any given point. He drew it until it's x-intercept (or root). From here he drew a circle where the normal to the curve was its radius and the root of the normal was its centre, hence the circle itself would be tangential to the curve at the given point.

[1] "I think therefore I am"

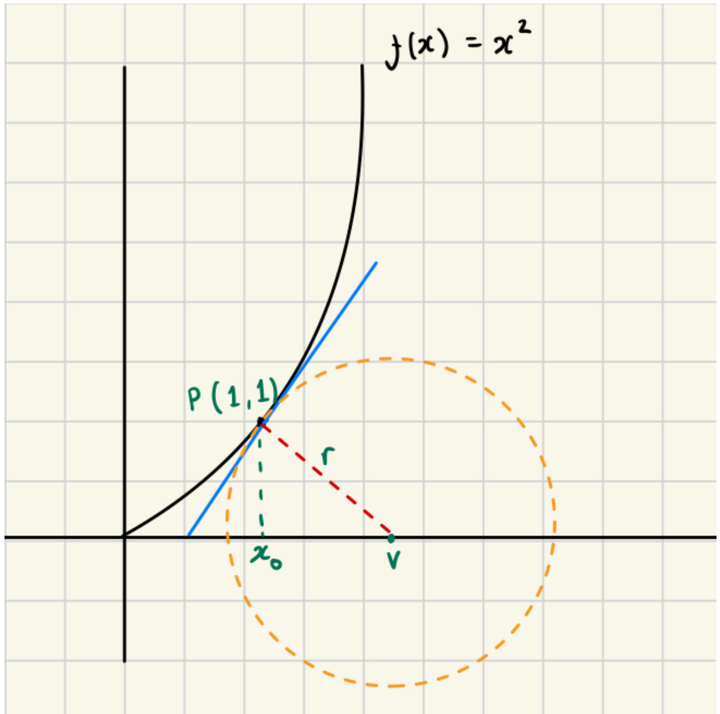

Figure 1

In Figure 1 you can see, in black, the curve that we would like to find the tangent to. Point *P*, in this case *(1,1)*,[2] is the point at which we want to find our tangent line. *r* is the normal to the curve at this point, which is in turn the radius of our circle. And in yellow you can see the circle, denoted by the general equation $(x-v)^2+y^2=r^2$ (where *V (v, 0)* is the centre of our circle.)

We aim to first find *v*, as from here we can find the gradient of the radius, which enables us to find the gradient of the tangent (the negative reciprocal of the gradient of the radius).

Some relatively simple but long maths can get us to this point.

Firstly, let us state the equations we know.

$$y = f(x)$$

$$r^2 - (x-v)^2 - y^2 = 0$$

Hence

[2] Note that by using P*(1, 1)* in certain areas the maths becomes easier, a luxury our 17th-century mathematicians would not have had.

$$r^2 - (x - v)^2 - f(x)^2 = 0$$

We know x_0 is a root of this polynomial, and it is even a repeated root as this is the only point where these two equations intersect[3]. So, $(x - x_0)^2$ must be a solution to this function, alongside another polynomial which we will denote as $g(x)$.

$$(x - x_0)^2 \cdot g(x) = 0$$

$$(x - 1)^2 \cdot g(x) = 0$$

$$(x^2 - 2x + 1) \cdot g(x) = 0$$

$$r^2 - (x - v)^2 - f(x)^2 = (x^2 - 2x + 1) \cdot g(x)$$

$$r^2 - (x - v)^2 - (x^2)^2 = (x^2 - 2x + 1) \cdot g(x)$$

$$r^2 - x^2 + 2vx - v^2 - x^4 = (x^2 - 2x + 1) \cdot g(x)$$

From observing both the right and left side, we can see that $g(x)$ must be a quadratic, as the highest order of x on the left side is an x^4. So, we can replace $g(x)$ with the standard form of a quadratic, $(ax^2 + bx + c)$.[4]

$$r^2 - x^2 + 2vx - v^2 - x^4 = (x^2 - 2x + 1)(ax^2 + bx + c)$$

From here we can use a method called comparing coefficients to find *a, b* and *c* and therefore also our *v*, which is our most crucial step in finding the gradient of the tangent line.

For the x^4 term:

$$ax^4 = -x^4$$

$$a = -1$$

For the x^3 term:

$$0x^3 = -2ax^3 + bx^3$$

[3] "Intersect" is used loosely here, as these equations never truly intersect and instead just meet at this point.
[4] Had our original function been of a higher difficulty than $f(x) = x^2$, this would be a much more complicated situation.

$$0 = -2(-1) + b$$

$$b = -2$$

For the x^2 term:

$$-x^2 = ax^2 - 2bx^2 + cx^2$$

$$-1 = -1 + -2(-2) + c$$

$$c = -4$$

For the x term:

$$2v = b - 2c$$

$$2v = 6$$

$$v = 3$$

We have now completed the first, and hardest step. We've found v! Therefore, we now know two coordinates of the radius, *P:* $(1,1)$ and *V:* $(3,0)$ so, we can easily find the gradient of the radius (m_r).

$$m_r = \frac{1-0}{1-3}$$

$$m_r = -\frac{1}{2}$$

From here we can find the gradient of the normal m_n (our tangent line) as it is the negative reciprocal of m_r.

$$m_n = 2$$

Finally, with this, we can complete our tangent line problem by finding the equation of the tangent!

$$y = 2x + c$$

$$1 = 2(1) + c$$

$$c = -1$$

$$y = 2x - 1$$

Clearly, Descartes' method works. However, that was a lot of work to get to our goal, even with a standard function ($f(x) = x^2$). So, you can only imagine the amount of work needed for a more complicated one. Additionally, as we need $y = f(x)$ in our first line of working out, our function has to be able to be written in this way (explicitly). So, his method would fail at any function that cannot be written this way, also known as an implicit function. Also, his method only works easily for polynomial functions.[5] Past these, the centre of the circle would not be placed on an axis, causing the maths associated to be considerably more difficult. Descartes received these critiques from many fellow mathematicians and responded in a very sensible and levelheaded way:

"**Monsieur Pascal has too much vacuum in his brain" - Descartes to Huygens**

"Don't send any more [critiques] because we already have plenty of scrap paper, which is all they are good for." - Descartes to Jean de Beaugrand

---

THE FOLIUM OF DESCARTES:

Descartes, being somewhat of his own biggest critic, then proposed a graph in 1638, where his method did not work (an implicit equation). This graph is called the Folium[6] of Descartes and is denoted by the equation $x^3 + y^3 - axy = 0$ where *a* is any constant. (Notice how this equation cannot be written in the form $y = f(x)$.)

[5] Remember, in the 17th century mathematicians were mostly investigating polynomials and not more abstract functions like $f(x) = e^x$ or $f(x) = \sin x$

[6] Leaf in Latin

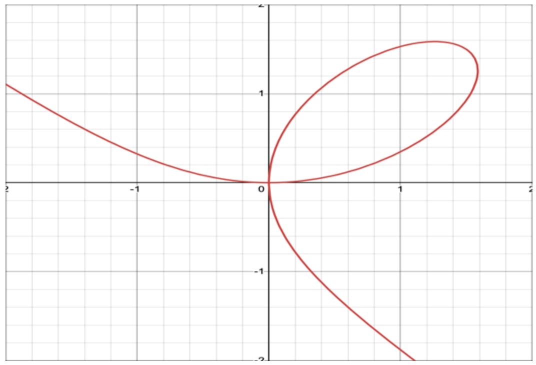

Figure 2

Descartes, now believing that finding the tangent to this graph at any general point was to be impossible, sent this equation to Pierre de Fermat challenging him to do what he could not. Yet what Descartes didn't consider was that Fermat was even more confident than the handful of other French mathematicians.

“It is impossible for me to be inferior to any other human being.” - Fermat.

PIERRE DE FERMAT:

Pierre de Fermat was a French lawyer who dabbled in *some* mathematics. Nevertheless, his recreational maths laid the foundations for many modern-day fields. Fermat is most noted for his “Last Theorem”, proven relatively recently by Andrew Wiles in 1993. Yet his steps towards discovering infinitesimal calculus are arguably his most influential discoveries. Such as his adequality method, his way of solving the tangent problem. Surprisingly he published this method first, c. 1636, but Descartes was only made aware of it after Fermat asked Mersenne[7] to send Descartes his method.[8]

Fermat started his method by drawing two vertical lines from the x-axis to the tangent, the first placed at our given point (In this case, once again *(1, 1)*) and the next placed at the

[7] Another prolific French mathematician who was mostly occupied by prime numbers.

[8] Alongside the message **“Perhaps having them put forward naked and without demonstration, they were not understood or they appeared too simple to M. Descartes, who has made so much headway and has taken such a difficult path for these tangents in his Geometry.”** (This is a savage burn in the world of maths).

point $(x + a\ ,y_1)$[9], where a is an arbitrary value. These lines can be seen in red and yellow in Figure 3.

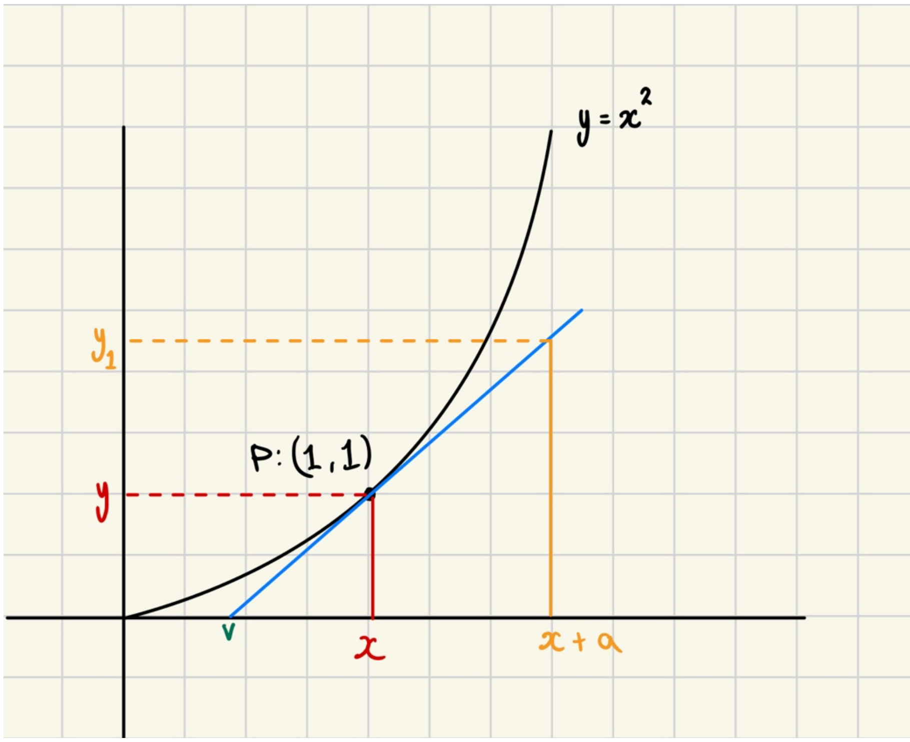

Figure 3

From here two mathematically similar triangles can be constructed. These triangles are made from the tangent line, the x-axis and the two lines we drew. As seen in Figure 4.

[9] In Fermat's original demonstration of this method, he uses the letter *e*, however Euler seems to have claimed that letter so for peace of mind we will use *a*.

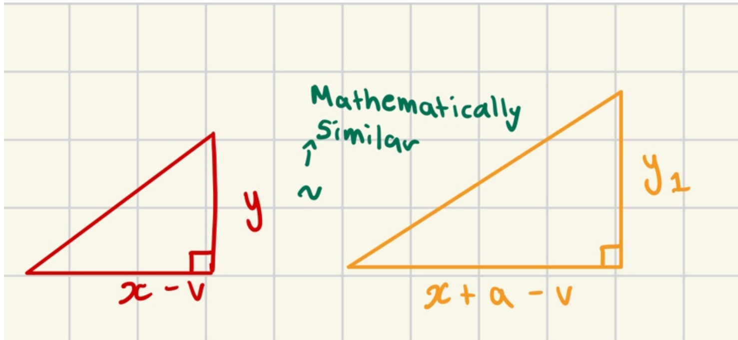

Figure 4

Mathematically similar means the angles in the shape are the same, and the sides of the shape are proportional. This gives us a neat starting point for our tangent line problem as the gradients are equal. Hence:

$$\frac{y}{x-v} = \frac{y_1}{x+a-v}$$

$$y_1 = \frac{y(x+a-v)}{x-v}$$

$$y_1 = y\left(\frac{x-v}{x-v} + \frac{a}{x-v}\right)$$

$$y_1 = y\left(1 + \frac{a}{x-v}\right)$$

The next important part of Fermat's method is understanding that on a cartesian plane, a general explicit curve can be expressed as, $y = f(x)$ but *any* general curve can be expressed in the form $F(x, y) = 0$. This just denotes that the curve is a function of *x* and *y* and (x, y) is a point on the curve. This means we can use this to generalize Fermat's method to every curve (including the Folium of Descartes).

So now we know:

$$y = x^2$$

$$y = f(x)$$

$$\Rightarrow f(x) - y = 0$$

$$F(x, y) = 0$$

$$F(x+a, y_1) = 0$$

$$\Rightarrow F\left(x+a, y\left(1+\frac{a}{x-v}\right)\right) = 0$$

From here Fermat's method requires some simple algebra and the bending of some mathematical rules, which we will consider later.

$$F(x, y) = f(x) - y$$

$$F(x, y) = x^2 - y$$

$$0 = x^2 - y$$

$$F\left(x+a, y\left(1+\frac{a}{x-v}\right)\right) = (x+a)^2 - y\left(1+\frac{a}{x-v}\right)$$

We know the left side of this equation equals 0, as we found out previously.

$$0 = (x+a)^2 - y\left(1+\frac{a}{x-v}\right)$$

$$0 = x^2 + 2ax + a^2 - y - \frac{ay}{x-v}$$

We also know that $x^2 - y = 0$:

$$0 = 2ax + a^2 - \frac{ay}{x-v}$$

$$(\div a)$$

$$0 = 2x + a - \frac{y}{x-v}$$

Now let $a = 0$ as if the two points were becoming infinitesimally closer making *a* negligible. (Here is where we bend the laws of maths; a little hypocritical coming from a lawyer.)

$$0 = 2x - \frac{y}{x-v}$$

We now have an equation for *v* in terms of only *x* and *y*. These *x* and *y* are given to us in the first place as these are just the coordinates of our given point. So, some simple substitution will get us to our *v*.

$$0 = 2(1) - \frac{1}{1-v}$$

$$2 = \frac{1}{1-v}$$

$$\frac{1}{2} = 1 - v$$

$$v = \frac{1}{2}$$

Now similarly to Descartes’ method, we have two points on the tangent line and can hence find its full equation. First by finding the gradient of the tangent:

$$m_t = \frac{1-0}{1-\frac{1}{2}}$$

$$m_t = 2$$

So, with this value, we can once again complete our tangent line problem.

$$y = 2x + c$$

$$1 = 2(1) + c$$

$$c = -1$$

$$y = 2x - 1$$

The adequality method works and is considerably more elegant than Descartes' method. Fermat poses a general equation, and his method can be used for all curves no matter whether they are implicit or explicit, hence why it works on the Folium of Descartes. However, the method requires that we divide by a, then we let a = 0. In maths, dividing anything by 0 is undefined and therefore this poses the question of whether Fermat’s method holds up at all.

What Fermat didn’t establish was that this method relied on limits, the same idea that Newton’s “first principle of differentiation” relies upon. Descartes was the first to point out the issues in the adequality method but as of their time, it was still considered the most revolutionary method. Thus, the ever-competitive Fermat was now happy.

"I promise you I am a lover of justice and peace." - Pierre de Fermat.

ISAAC NEWTON:

Mathematics is constantly evolving, so of course mathematicians were not entirely content with Fermat's adequality method. There were far too many issues with it and, although more elegant than Descartes' method, it didn't pose an entirely generalized equation which was our goal in the tangent line problem. However, the ideas that came from Fermat's method were essential. Had Fermat been more precise about his definitions would we be calling him the father of differential calculus? Well, we aren't. Instead, we reward this title to Isaac Newton (yes, the apple guy), an English polymath[10] who was involved in many fields of maths, physics and theology.

Newton's method to find the tangent line was very similar to Fermat's in the idea that it required limits, yet Newton defined what a limit was much more precisely. This allowed him to construct a considerably more elegant general equation. It is, however, argued that Newton was unsure of what a limit was himself and his definition was vague and lacked depth[11], but for the sake of this essay we will assume that he understood the maths he was doing. He published his method in "Method of Fluxion" which he completed in 1671 but was only published after his death in 1736.

He started his method very similarly to Fermat, drawing a curve with the general equation $y = f(x)$. Unfortunately, we have regressed slightly with this method, and it can only work for explicit equations. From here he drew two lines from the curve down to the x-axis, one at our given point and the other at a point $x + h$ where h is an arbitrary value. (Notice the similarities between Newton's method and the adequality method.)

[10] Someone whose knowledge spans multiple subjects

[11] Pourciau's Paper "Newton and the Notion of Limit" explores this interestingly
Click Here for Paper

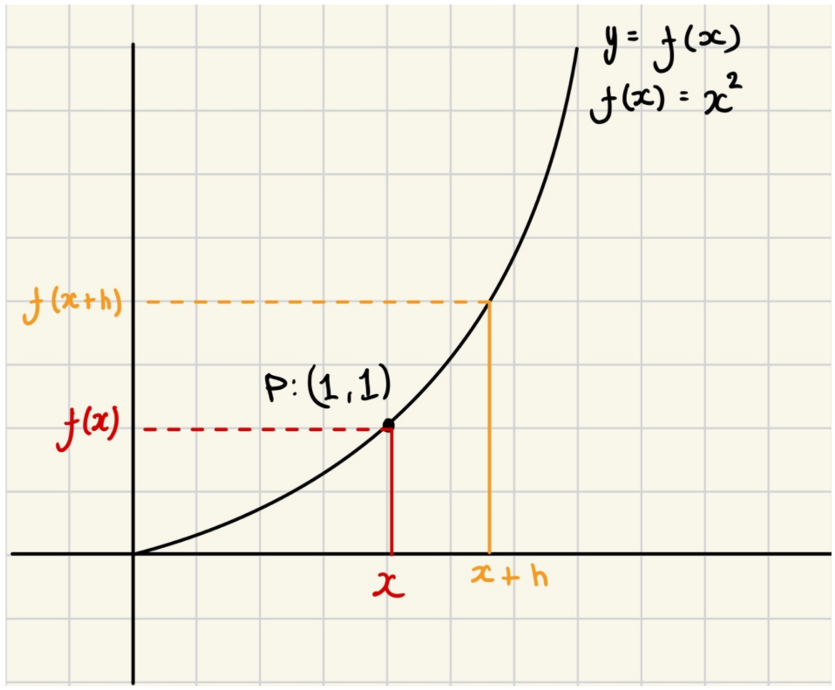

Figure 5

As you can see in Figure 5, the attributed y coordinates are $f(x)$ and $f(x+h)$. From here we can use a simple slope formula.

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x + h - x}$$

Then we introduce limits. As *h* gets smaller and smaller to the point where it is nearly 0, the gradient of the two lines, found by the equation above, will be exactly the gradient of our curve. So, in terms of the equation -

$$\frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$\frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$\frac{\Delta y}{\Delta x} = \lim_{h \to 0} 2x + h$$

$$\frac{\Delta y}{\Delta x} = 2x$$

This gives us that the gradient of the curve $y = x^2$, for any x coordinate, is $2x$ and therefore from here we can find the final equation of the tangent.

$$m_t = 2(1)$$

$$m_t = 2$$

$$y = 2x + c$$

$$1 = 2(1) + c$$

$$c = -1$$

$$y = 2x - 1$$

The origins of this method are up for much debate as Leibniz and Newton both claim to have defined it first. Yet what is definite is that this method was an extension of Fermat's method. Additionally, this method can only be used with explicit functions, yet it is used to prove the chain rule of differentiation by letting $f(x) = g(h(x))$.[12] As well as, being used to prove the product rule of differentiation by letting $f(x) = g(x)h(x)$.[13] From here we can implicitly differentiate any function. So, we can see that this formula is by far the most elegant solution we have to the tangent line problem.

"I had the hint of this method [of fluxions] from Fermat's way of drawing tangents, and by applying it to abstract equations, directly and invertedly, I made it general." - Isaac Newton[14]

---

[12] See derivation here - https://mathsfromnothing.au/derivation-of-the-chain-rule-from-first-principles/
[13] See derivation here - https://unacademy.com/product-rule-formula-using-the-first-principle/
[14] As quoted by Louis Trenchard More (1934)

CONCLUSION:

No matter how much you fight it, mathematics is a collaborative subject. Mathematicians bounce ideas off each other to improve our understanding of the world and its equations. The rivalry between Fermat and Descartes may have seemed petty and childish from afar when in reality it was much needed for the evolution of the subject. This is why it's so hard to dictate who truly was the first to differentiate. Descartes' method was needed to challenge Fermat. Fermat's method was needed to inspire Newton. Equally, Newton's pursuit encouraged Leibniz.

Calling Newton the "father of differential calculus" could be correct, but dismissing the work that Descartes, Fermat and Leibniz conducted would be foolish. Instead, I invite you to construct your own opinion on where the true origins of differentiation lie, whether that be Touraine, Toulouse, Lincolnshire or Leipzig.

REFERENCES:

All diagrams are my own.

Mathematical Ideas inspired by "Another Roof" - https://www.youtube.com/watch?v=xKfEmbWBgvM&t=563s

Quotes -

- https://www.youtube.com/watch?v=znSrogfeRkM
- https://en.wikiquote.org/wiki/Pierre_de_Fermat
- https://www.bookey.app/quote-author/pierre-de-fermat

ACKNOWLEDGEMENTS:

Thank you to Matt Parker and his book "Love Triangle" for encouraging me to write about maths entertainingly!