Finding the fastest path to cross a river.

Introduction

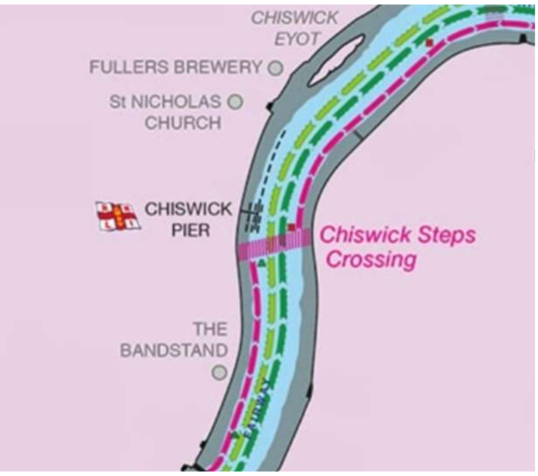
A river problem

As a rower on the river Thames, I have to respect certain navigation rules. When going with the stream, you are required to stay in the middle of the river, where the current is the fastest. This leaves the areas near the banks of the river for those going upstream, where the current is not as strong, and so easier to row against.

Every few kilometres, there are navigational buoys which indicate to those going upstream that they must cross over to the other side of the river to stay in the inshore zone, or the inside of a bend, where the current is the slowest.

I train for rowing 12 times a week, of which 4 or 5 are on the water. During the long and relatively easy sessions, which are always on the same stretch of water, from Putney to Chiswick, it can get quite boring. So, I distract myself with mathematical problems to pass the time, such as estimating how many strokes left to take, or exactly what percentage of the session has passed already.

As someone not usually responsible for steering the boat, that being the job of the coxswain, I often wonder during these sessions if there is a faster way to cross the river than the usual straight line taken, perhaps spending a bit longer on the initial side of the river to avoid spending time in the middle, which is slower.

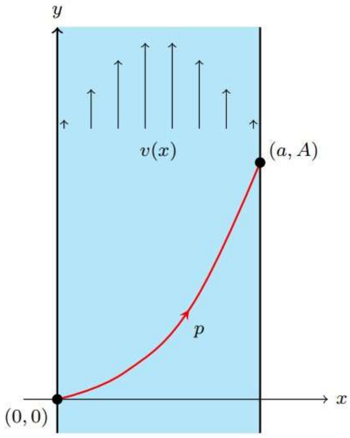


Map of River Thames provided by RNLI

This seemed to me at first a very simple calculus optimisation problem, with not too many complications. However, the variable current speed seemed tricky, but it was the idea that a boat can take one of an infinite number of paths between the two points that seemed daunting. Finding the local minimum or maximum of a single- or multi-variable function is simple enough: just take the derivative and find its zeros. But this approach would not work with functions. Is there such a thing as a derivative but over functions?

This is the job of the calculus of variations, an area of mathematics which deals with functions as inputs, rather than single variables.

Definition



We will take the banks of the river to be the y-axis and the line defined by $x = a$. The start and end points of the path taken p are $(0, 0)$ and (a, A) respectively, where a and A are both positive. The speed profile of the current is defined by the function $v(x)$. We will assume that both $p(x)$ and $v(x)$are smooth continuous functions. The boat travels at a constant speed c.

Terminology and Theorems

The analysis later will use terminology and theorems[1] which might be useful to define and explain below.

A **functional** $J[f]$ is a map from functions f to the real numbers[2]. More intuitively, it is a "function" whose domain is the set of all functions and codomain is the set of reals, in much the same way that a regular function operates on reals to output reals; a 'function of a function'.

A **Lagrangian** $L(x, y, y')$ is a function which acts upon not only the input, but also the function and its first derivative, useful in many instances across maths and physics, as it represents the full state of a system.

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[1] See appendix for proofs
[2] mathworld.wolfram.com/Functional.html

A **partial derivative** $\frac{\partial y}{\partial x}$, as opposed to a **total derivative** $\frac{dy}{dx}$, is one which, in a multivariable expression, only takes the derivative with respect to x, and considers all others to be constant with respect to x. Importantly, these derivatives can also be of the form $\frac{\partial S}{\partial f}$, or derivatives of a functional with respect to the function. For convenience we will use f' to denote the derivative of the function f with respect to x. For example, for a function $f(x, y) = x^2y + \sqrt{y}$, $\frac{\partial f}{\partial x} = 2xy$ because we consider y to be a constant.

Leibniz's integral rule[3] states that for a continuous function f, where $\frac{\partial f(x,t)}{\partial x}$ is also continuous,

$$\frac{d}{dt}\int_a^b f(x, t)\, dx = \int_a^b \frac{\partial}{\partial t} f(x, t)\, dx$$

Put simply, it allows us to switch the derivative and integral in order to differentiate under the integral sign.

The **multivariable chain rule**[4] is the extension of the single variable chain rule it states that for real valued function y,

$$\frac{\partial y}{\partial x} = \sum_{k=1}^{n} \frac{\partial y}{\partial u_k}\frac{\partial u_k}{\partial x}$$

Conceptually, this represents the fact that any small change along x may result in a change along the related variables u_1 through to u_n, all of which result in a change in y.

Lagrangian interpolation[5] is rule which states that for a set of points (x_1, y_1), … , (x_{n+1}, y_{n+1}), there is a unique polynomial $p(x)$, which can be defined as

[3] mathworld.wolfram.com/LeibnizIntegralRule.html
[4] math.libretext.org
[5] mathworld.wolfram.com/LagrangeInterpolatingPolynomial.html

$$p(x) = \sum y_1 \frac{(x-x_2)\ldots(x-x_n)}{(x_1-x_2)\ldots(x_1-x_n)}, \text{ taken cyclically over all points.}$$

Essentially, the fraction part of the sum is simply one which is equal to 1 for the current input, and 0 for all other points. We then multiply by the output corresponding to the current input. This ensures that the polynomial passes through all the points.

Deriving the functional

In this section, our aim is to find the expression of the functional $T[p]$, which will be defined as the time taken to cross the river from point $(0, 0)$ to point (a, A). As is common with variational calculus, it will be of the form

$$T[p] = \int_0^a L(x, p, p')\, dx, \qquad p(0) = 0, p(a) = A$$

We first assume that the vertical and horizontal components of c, which we will call c_x and c_y, are both positive. This makes sense as the boat has to move towards (a, A) rather than away from it at any point in time.

Splitting c up, according to the Pythagorean theorem,

$$c^2 = c_x^2 + c_y^2$$

Relative to the water, the velocity of the boat can be parametrised as a vector $\begin{pmatrix} c_x \\ c_y \end{pmatrix}$.

But accounting for the current, which simply adds on to the vertical component, the velocity relative to the bank becomes $\begin{pmatrix} c_x \\ c_y + v \end{pmatrix}$.

Because velocity is simply the change in position, for the path p,

$$\frac{dp}{dx} = \frac{v + c_y}{c_x}$$

Rearranging for c_y,

$$c_y = c_x \frac{dp}{dx} - v$$

We know that $speed = \frac{distance}{time}$, so $t = \frac{d}{v}$. We can then consider the speed relative to the x-axis, to give the time T, with dx the small change in position and c_x the speed,

$$T[p] = \int_0^a \frac{dx}{c_x}$$

It remains to express c_x in terms of elements and variables defined earlier, so that the expression depends only on c, v, x, y and y' and the functional is defined. Substituting the expression of c_y into $c^2 = c_x^2 + c_y^2$, we get

$$(p'c_x - v)^2 + c_x^2 = c^2$$

Expanding,

$$p'^2 c_x^2 - 2p'vc_x + v^2 + c_x^2 - c^2 = 0$$

We now get a quadratic equation in terms of c_x,

$$\left(1 + p'^2\right)c_x^2 - 2p'vc_x - (c^2 - v^2) = 0$$

Therefore, the solutions for c_x are

$$c_x = \frac{p'v \pm \sqrt{(p'v)^2 + (c^2 - v^2)\left(1 + p'^2\right)}}{\left(1 + p'^2\right)}$$

We assume that $c > v$, as is the case in rowing, and we have c_x being positive, so we take the positive solution. We can then rationalise the numerator, because it will become the denominator in the functional, multiplying through by $p'v - \sqrt{(p'v)^2 + (c^2 - v^2)\left(1 + p'^2\right)}$.

$$c_x = \frac{p'v + \sqrt{(p'v)^2 + (c^2 - v^2)\left(1 + p'^2\right)}}{\left(1 + p'^2\right)}$$

$$= \frac{(p'v)^2 - (p'v)^2 + (c^2 - v^2)\left(1 + p'^2\right)}{\left(1 + p'^2\right)\left(\sqrt{(p'v)^2 + (c^2 - v^2)\left(1 + p'^2\right)} - p'v\right)}$$

The $(p'v)^2$ cancel out, as do the $\left(1 + p'^2\right)$.

$$c_x = \frac{c^2 - v^2}{p'v - \sqrt{(p'v)^2 + (c^2 - v^2)\left(1 + p'^2\right)}}$$

The final time taken by a path p is therefore defined by

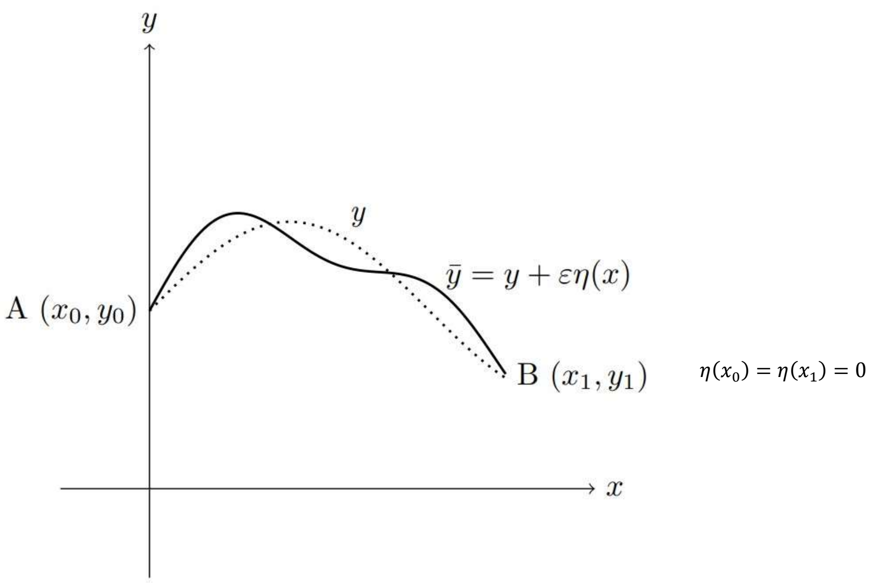
$$T[p] = \int_0^a \frac{\sqrt{(p'v)^2 + (c^2 - v^2)\left(1 + p'^2\right)} - p'v}{c^2 - v^2} dx = \boxed{\int_0^a \frac{\sqrt{c^2\left(1 + p'^2\right) - v^2} - p'v}{c^2 - v^2} dx}$$

The Euler-Lagrange equation

Now, with a defined functional, we have to find its stationary path, or extremal, that is, to minimise it.

We can approach this in much the same way as we do with single-variable functions. The local minima and maxima of a function are when the derivative is equal to zero, but also the point at which a small change in x (dx), will result in a change in an output further away in the same direction from your previous output.

Given a functional $J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx$, where the function $y(x)$ optimises $J[y]$, the 'small change' is no longer a single value, but rather a function, which by convention we denote $\varepsilon\eta(x)$. We can then define a new function $\bar{y} = y + \varepsilon\eta(x)$, along the same boundary conditions, so that they are equal at A and B.



We can then observe that $J[\bar{y}]$ simply becomes a function of ε, $J(\varepsilon)$. This is due to y being a set function, and $\eta(x)$ being an infinitesimally small functional variation (just like dx is for functions).

This allows us to make a series of logical steps. Firstly, when $\varepsilon = 0$, $\bar{y} = y$, because the change in the function disappears. Now, since y is an extremal of J, ε is a local minimum or maximum of J.

$$J'(\varepsilon = 0) = 0$$

Expanding $\frac{dJ}{d\varepsilon}$,

$$\frac{d}{d\varepsilon}\int_{x_0}^{x_1} F(x, y, y')\, dx$$

From Leibniz's integral rule, we can simplify it to

$$\frac{dJ}{d\varepsilon} = \int_{x_0}^{x_1} \frac{\partial F}{\partial \varepsilon}\, dx$$

Applying the multivariable chain rule, where the other variables aside from F are $\bar{y}$ and $\bar{y}'$, this integral becomes

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \varepsilon} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \varepsilon}\, dx$$

We know that $\bar{y} = y + \varepsilon\eta(x)$. So, by definition, $\frac{\partial \bar{y}}{\partial \varepsilon} = \eta(x)$, as that is the infinitesimally small change in $\bar{y}$, almost like a 'functional gradient', with y as a constant. Consequently, $\frac{\partial \bar{y}\prime}{\partial \varepsilon} = \eta'(x)$. The expression becomes

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial \bar{y}}\eta + \frac{\partial F}{\partial \bar{y}'}\eta'\, dx$$

Using integration by parts, we can evaluate the second term. We can take $u = \frac{\partial F}{\partial \bar{y}\prime}$, and $dv = \eta'$. So, $du = \frac{d}{dx}(\frac{\partial F}{\partial \bar{y}\prime})$ and $v = \eta$. Laying it out,

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial \bar{y}'} \eta' \, dx = \left[\frac{\partial F}{\partial \bar{y}'} \eta\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right) dx$$

However, since $\eta(x_0) = \eta(x_1) = 0$, the uv term goes to zero when x_0 and x_1 are substituted.

From all this, we can see that

$$\frac{dJ}{d\varepsilon} = \int_{x_0}^{x_1} \eta \left[\frac{\partial F}{\partial \bar{y}} - \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right)\right] \eta \, dx$$

To find an extremal, the function J, $\frac{dJ}{d\varepsilon}$ must be equal to zero. This is true when the integrand is equal to zero. However, η is an arbitrary function, and so the integral can only be equal to 0 if the other part is. Therefore, a stationary path must satisfy

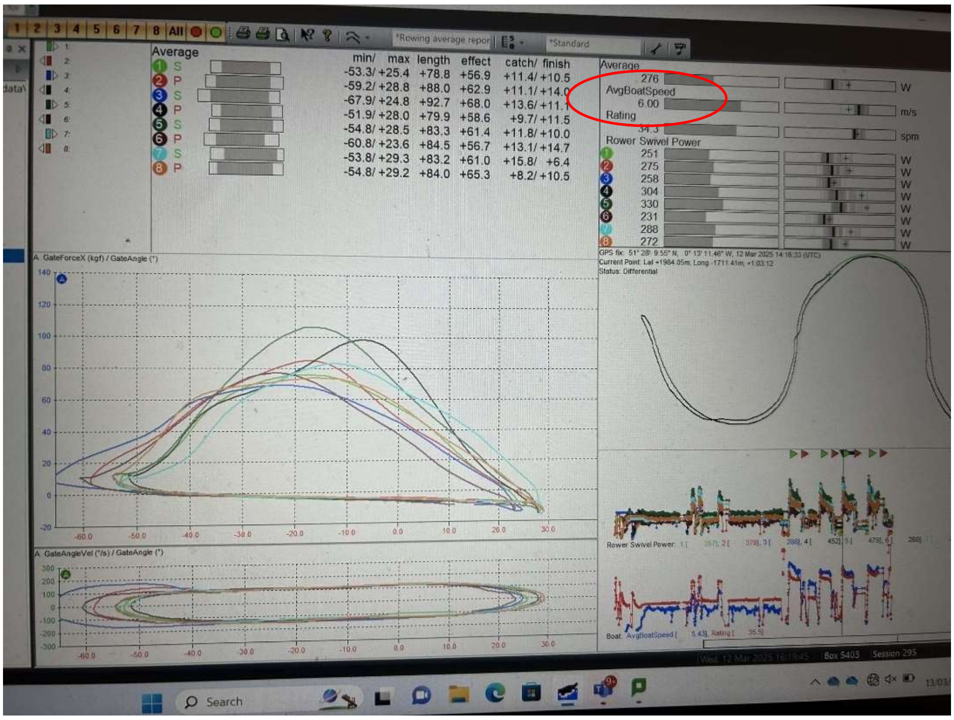
$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \qquad y(x_0) = y_0 \;\; y(x_1) = y_1$$

This is the Euler-Lagrange equation, which acts as Fermat's theorem for functionals rather than functions, in that, if satisfied, it will give the input minimising the function. It is one of the important theorems of the calculus of variations

Calculating the path

River current

Before substituting our functional into the Euler-Lagrange equation, we must come up with values for c and $v(x)$, as well as for the boundary condition (a, A).



Picture of the telemetry system installed on the boats, recording power and speed.

From rowing telemetry systems installed on our boats, we can approximate our speed on still water to be 5m/s (accounting for stream), which we will take as our value for c.

Using measurements from Google Earth, at the Chiswick crossing, the river is 70m wide, and the crossing points are 50m apart. Usually, to take a smoother line, coxswains will end the crossover further down the river, not crossing over directly, but rather going

View of the Thames from above, at the Chiswick crossing.

diagonally across. We will take this distance to be 100m. The coordinate (a, A) is therefore $(50, 100)$ for this example.

We will assume that the current is at its fastest right in the middle of the river and that it is significantly slower near the banks, as is the case for many straight rivers. To simplify, we will take the current speed to be 0m/s right at the banks of the river. The river Thames has a current speed of around 1m/s in the middle[6]. So, we must find a function $v(x)$ whose graph passes through $(-10, 0)$ for the left bank, $(60, 0)$ for the right and $(25, -1)$ for the maximum speed in the middle. The speed is negative because the boat is moving in the positive direction, against the stream, which is therefore negative.

[6] According to the-river-thames.co.uk/weather.htm

We could choose any function to go through these points, but we will opt for a quadratic function for simplicity and as it is easy to fit to the data.

We can use Lagrangian interpolation to find the quadratic which goes through these three points. The elements for the roots become zero, so,

$$v(x) = -\frac{(x+10)(x-60)}{(25+10)(25-60)} = \frac{1}{1225}x^2 - \frac{2}{49}x - \frac{24}{49}$$

Solving the differential equation

What remains is substituting into our functional and solving using the Euler-Lagrange equation.

Our functional becomes

$$T[p] = \int_0^{50} \frac{\sqrt{25+25p'^2 - v^2} - p'v}{25 - v^2} \qquad p(0) = 0 \;\; p(50) = 100$$

Analysing this functional, we can observe that there is no p term; that it is only in terms of x and p'. This simplifies our task, because the first term of the Euler-Lagrange equation, $\frac{\partial F}{\partial p}$, where F is the integrand, becomes 0. This is due to all other variables being considered constants, and the derivative of a constant is always 0.

The equation is then simplified to

$$\frac{d}{dx}\left(\frac{\partial F}{\partial p'}\right) = 0$$

Integrating both sides, to get rid of the total derivative, we get, for an arbitrary constant C,

$$\frac{\partial F}{\partial p'} = C$$

Splitting our integrand into two fractions, we can take their partial derivatives with respect to p', where v acts as a constant.

$$\frac{\partial}{\partial p'}\frac{-p'v}{25-v^2} = -\frac{v}{25-v^2}$$

$$\frac{\partial}{\partial p'}\frac{\sqrt{25+25p'^2-v^2}}{25-v^2} = -\frac{25p'}{(25-v^2)\sqrt{25+25p'^2-v^2}}$$

This gives us the equation

$$\frac{v}{25-v^2}+\frac{25p'}{(25-v^2)\sqrt{25+25p'^2-v^2}} = C$$

Bringing the first term to the other side, then multiplying by $25 - v^2$,

$$\frac{25p'}{\sqrt{25+25p'^2-v^2}} = (25-v^2)C - v$$

Squaring both sides,

$$\frac{625p'^2}{25+25p'^2-v^2} = ((25-v^2)C-v)^2$$

Multiplying both sides by $25 + 25p'^2 - v^2$, then rearranging to isolate p',

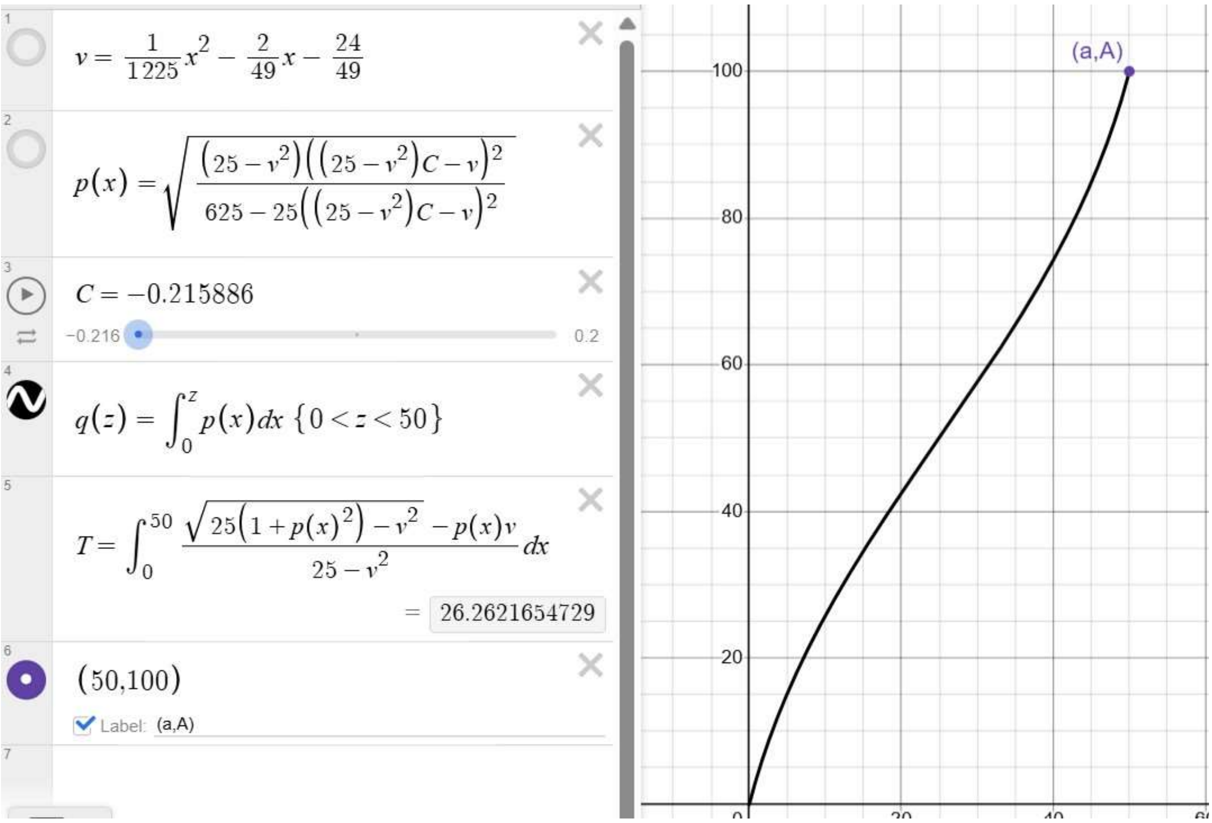
$$\left(625-25\left((25-v^2)C-v\right)^2\right)p'^2 = (25-v^2)\left((25-v^2)C-v\right)^2$$

$$p' = \sqrt{\frac{(25-v^2)\left((25-v^2)C-v\right)^2}{625-25\left((25-v^2)C-v\right)^2}}$$

To find p, we must integrate both sides of the equation. Unfortunately, the expression on the right-hand side is not integrable manually and does not have an elementary closed form expression. We can, however, solve this numerically. The path is then

$$p = \int \sqrt{\frac{(25 - v^2)\left((25 - v^2)C - v\right)^2}{625 - 25\left((25 - v^2)C - v\right)^2}}\, dx$$

We can use Desmos to approximate what C is. In order for the boundary conditions to be satisfied, I found that $C \approx -0.215886$. (Here the p' is named p and p is the function q).



Desmos graph of river crossing path

Once we have found C, we can substitute the p back into the functional T. We find the time to cross the river to be 26.26 sec, and the optimal path is the curve displayed.

We can also check whether a path minimising distance travelled, a straight line (a variational problem in and of itself) would be faster. Replacing p' with 2, so that $p(x) = 2x$, we find the time taken to be 26.38 sec, which is marginally slower.

Conclusion

This is an exact solution to the problem we set out in the introduction. Unfortunately, we cannot use this to our advantage for rowing applications, although it could be used for someone swimming across a river. We made many simplifications along the way.

Firstly, in order to navigate properly, the boat needs to be parallel to the bank when going into the crossover. The above trajectory, when added to the path before and after, would require the boat to turn instantaneously, which is obviously not possible, or would lead to a significant slowdown. This can be remedied using Neumann boundary conditions, which are conditions imposed alongside the differential equation to impose a value for the derivative at the boundaries of the problem (. Here, we would calculate the needed derivative of the path such that it would smoothly connect with the leadup to the crossover. This problem is, however, beyond the scope of this essay.

Secondly, a rowing boat is steered using a fin. So, any turns, especially sharp, increase drag, and cause a drop in boat speed. This is quite a significant factor when choosing a path to follow on the water. Solving for such a constraint would cross into the realm of computer simulation, as the exact decrease in speed is hard to quantify and there are no equations to describe this kind of behaviour.

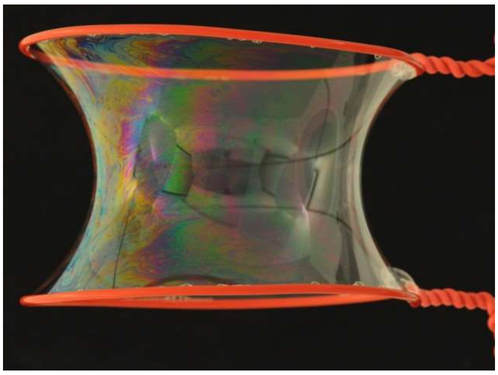
Finally, we assumed that the boat was a single point, and that the speed parameters instantly changed to account for the current speed. However, in reality, a standard 8+ boat is 20m long, with a very small cross-sectional area but a large surface area if observed from the side. Consequently, the effects of the current would be much more pronounced if the boat were perpendicular to the river than if it were parallel. Similarly, many factors go into how much the stream actually impacts boat speed, from the boat shape, as well as how gradually a boat slows down when encountering faster current. Once again, these issues are hard to resolve purely mathematically.

Calculus of Variations

The branch of mathematics and the techniques used in the solution to this problem are applicable to a wide range of questions.

Problems of a similar kind include the distance between two points along a path, with functional $J[y] = \int_a^b \sqrt{1 + y'} dx$, and the brachistochrone problem, which aims to find the curve along which a bead would slide down in the fastest time under the effect of gravity. The latter has functional $J[y] = \int_a^b \frac{\sqrt{1+y'}}{\sqrt{2gx}} dx$. The shape of the resulting curves for these problems, when the Euler-Lagrange equation is applied, are a straight line joining the two points for the first, and a cycloid for the second.

The multivariable equivalents to this problem can be found in surfaces of least area, or minimal surfaces, which are formed by soap film, for example.



Soap film assuming a minimal surface, given boundary conditions (the rings)

More broadly, it is closely related to physics, where, in the field of optics, Snell's Law dictates how light moves through different mediums. In fact, this simple equation arises from Fermat's principle, which dictates that light will travel the path which locally minimises time between two points.

Also, it is an extremely important tool in Lagrangian mechanics, which defines *action* as Kinetic energy – Potential energy. Equations derived from the calculus of variations govern how bodies move and interact, as action is at all times minimised, according to the Principle of Least Action. It is central to fields such as general relativity and quantum mechanics.

Appendix

Proof of Leibniz's integral rule

If we take the derivative of an integral in x with respect to t,

$$\frac{d}{dt}\int_a^b f(x,t)\,dx$$

We can apply the definition of a derivative to the integral,

$$\frac{\int_a^b f(x,t+\Delta t)\,dx - \int_a^b f(x,t)\,dx}{\Delta t}$$

Using the Taylor series to represent the first term we get

$$\frac{\int_a^b f(x,t) + \frac{\partial f}{\partial t}\Delta t\,dx - \int_a^b f(x,t)\,dx}{\Delta t}$$

The two $f(x,t)$ cancel out, as do the Δt term after that. We are left with the following expression, which is just the derivative and integral switched around.

$$\int_a^b \frac{\partial}{\partial t} f(x,t)\,dx$$

References

For proof of the Euler-Lagrange equation, which I made more intuitive,

J. Figueroa-O'Farrill, *Brief Notes on the Calculus of Variations*, Univ. of Edinburgh, p7

S. Noubir, The Euler-Lagrange Equation, p7

For general understanding and definitions,

Introduction to the Calculus of Variations, Open University

Desmos graph: www.desmos.com/calculator/eyfv2dml18

Self-made LaTeX graphs: River diagram, Euler-Lagrange graph