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# The problem that broke the world of mathematics.

## Introduction

Throughout history, humans have made mistakes in their lives, many of which are just small errors, such as a miscalculation or a small grammatical error. Although there are some mistakes that can lead to dire consequences, for instance, the design flaws in the Titanic causing it to sink, the mismanagement of the nuclear power plant in Chernobyl leading to the evacuation of 68,000 people, or even the England 2024 Euros final football tactics that now makes it nearly 60 years without a trophy. However, that made me ponder the question: has there been a similar problem that shook the world of mathematics?

It turns out that this indeed happened to the world of mathematics in 1901. Karl Frege and Bertrand Russell had spent years building the foundations of mathematics, creating statements that are true so that mathematicians in the future could create new ideas built upon them. Then in May, Russell discovered a contradiction that would erase years of work on the foundations of mathematics and sent this problem to Karl Frege. After Frege read the letter, the paradox was so disastrous to his works and the world of mathematics that he was hospitalised. In this essay I am going to talk about Russell's Paradox as well as its impact on mathematics and even philosophy and how mathematicians tried to solve and ultimately avoid the contradiction of set theory that it creates.

## The creation of sets.

Firstly, what is a set? A set, in simple terms, is a collection of items, and these items are called members or elements. A way to show that things are a set is to put elements in these brackets {}. If you wanted to create a set of all the fishes in the universe, you would create a set in the form {x: x is a fish}, which in words is a set of x where x is a fish.

Sets were first used by Georg Cantor to prove that some infinities are larger than other infinities. He first proved that you could have 2^n subsets (a set in a set) in a set with n elements in a set. For example, if a set has elements A, B, and C, you can create 8 subsets from this set: A, B, C, AB, BC, AC, ABC, and a null set. Now you can create a set made up of those subsets, and this set will have more elements than the previous one (in the example, it will contain 8 elements). If we create a set with all natural numbers (which is an infinite number of elements) called N0, then we can create another set which consists of all the subsets of N0. This new set will have more elements than in N0, and since both sets have an infinite number of elements, we can say that some infinities are larger than others.

## Naive set theory

Before Cantor used sets to prove that infinities are larger than other sets, he had to establish rules that a set obeys.

Rule 1. Axiom of unrestricted comprehension. Anything can be in a set, from the number 37 to vintage brass compasses, from anything you can imagine in a set or even unimaginable things in a set.

Rule 2. Axiom of extensionality. The set is defined by what is inside the set. Set A is equal to set B if and only if they have the same members.

Rule 3. The order of the set does not matter; this is somewhat linked to the 2nd rule. It doesn't matter where the elements are in a set; it is just what is in the set that determines what set it is. For instance, {1, 2} is the same as {2, 1}.

Rule 4. Repeats don't change the set. For example, the set {1, 2, 2} is the same as {1, 2}.

Rule 5. The descriptions in the set do not matter. For example, the set that contains only Lionel Messi and the set that contains the greatest footballer in the world is the same set containing the 1 element: Lionel Messi

Rule 6. A union of any 2 or more sets is a set. You can combine 2 sets to create a new set. For example, you can combine the set of all the kings and the set of all the queens to create the set of all the kings and queens. This links to Rule 1 since you can add anything in a set.

Rule 7. Any subset is a set. A subset is a set that contains some of the items in a set. Once again, since you can add anything into a set, a subset can therefore be a set.

Rule 8. A set can contain 1 member. This set is also known as a singleton set.

Rule 9. You can have a set with no members. This is known as an empty or null set and is shown as {} or Ø.

Rule 10. You can have a set of sets. For example, the set {{1,2}, {3,4}} contains the sets {1,2} and {3,4}, which shows us that you may have a set of sets.

Rule 11. A set can contain themselves. This comes from rule 1: if you can create any set, then why not create a set containing itself? For example, let's say set A is {x: x is a set} since set A is a set. It should be in set A.

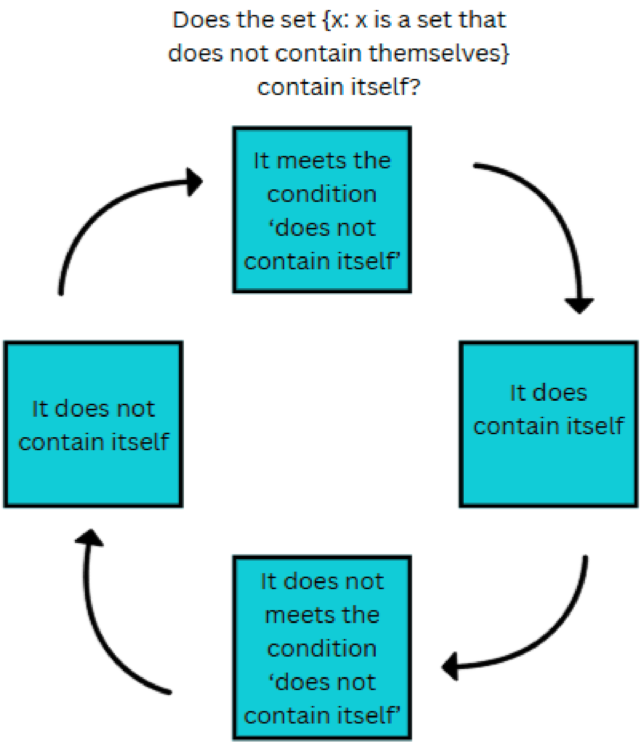
## The Paradox

In 1901 Bertrand Russell was just thinking about Rule 11, trying to consider sets that contain themselves and sets that don't contain themselves. The sets that do not contain themselves are sets like {3}, {x:x is a fish} and {x:x is a singleton set}. The set {3} cannot contain itself, as it only holds the element 3; the set of all fish only contains fishes, not a set like {x:x is a fish}. Finally, in the set {x:x is a singleton set}, there are many singleton sets, so the set {x:x is a singleton set} cannot be a singleton set itself, so it does not contain itself. Examples of sets that do contain themselves are {x:x is a set} and {x:x has been mentioned in this essay}. He then created 2 sets: the set of sets that contains itself (also denoted as {x:x is a set that contains itself}) and the set of sets that do not contain itself (also denoted as {x:x is a set that does not contain itself}). This is when he realised that there was a huge problem.

Does the set {x:x is a set that does not contain itself} contain itself?

If we look at the 2 possibilities, if it does contain itself, then it should meet the condition that it does not contain itself, so it doesn't contain itself. If we look at the alternative, it doesn't contain itself, then it can fit into the criteria; it does not contain itself, so it should contain itself.

So, if it doesn't contain itself, it contains itself, and if it does contain itself, it doesn't contain itself. Now we have a contradiction, and therefore something has gone wrong in naive set theory.




The diagram above shows how Russell's Paradox creates a contradiction in set theory.

## Consequences of the paradox

In the late 1700s a philosopher called Immanuel Kant proposed that mathematics was a construction of the human mind, which would make mathematics subjective if that idea was true. This contrasted with Bertrand Russell and Carl Frege's thought of logicism to make maths be objective. They theorised that mathematics is a branch of logic and that the simple forms of mathematics (arithmetic) can be broken down into first-order logic consisting of axioms (statements that are regarded as being self-evidently true) and set theory. They further proposed that numbers are sets and that the number 1 is the set of all singleton sets and 2 is all the sets with 2 members. This made set theory an integral part of mathematics, as it built the main foundations of mathematics.

However, as I have shown you, Russell's paradox proves that something has gone wrong in set theory, and if set theory has a flaw, then the rest of mathematics was in doubt, as most of it relied on set theory. This also meant Frege's and Russell's years of work on making foundations of mathematics potentially useless, and now the rules of set theory and foundations of mathematics had to be rewritten.

## Philosophical responses to the paradox

Russell's paradox not only had a mathematical response but a philosophical response as well. The 2 main philosophical viewpoints that tackle Russell's paradox are constructivism and intuitionism.

Constructivism is the belief that it is necessary to construct a specific example of a mathematical object to prove that something exists. Some mathematical proofs aren't classified as constructive. For example, the proof of contradiction is nonconstructive since you assume the mathematical object's existence (to later show its contradiction). Therefore, you are not 'finding the object', meaning that a constructivist might reject the proof. The constructivists would be rejecting Russell's paradox because the idea of a set of all the sets that does not contain themselves assumes that there are an infinite number of sets which cannot be 'constructed'.

Created in 1912 by L.E.J. Brouwer, intuitionism is the belief that mathematics is from the human mind rather than the discovery of fundamental principles. Similarly to constructivism, they also do not accept the proof of contradiction and that mathematical objects should be constructed rather than proven to exist. An intuitionist may reject some elements of logic since it did not come from the human mind (to make sure that the fact is intuitionistically true). The intuitionists would argue that Russell's paradox shows the limitations of formal logic and therefore would reject the set rules that cause the paradox.

Both intuitionism and constructivism criticised classical logic and set theory, and even though they did not directly find a solution to the paradox, it highlighted the flaws in logic as well as the difficulties caused by self-referential statements, like how sets could contain themselves.

## Solving the paradox

The main solution to the paradox was to get rid of rule 11: sets can contain themselves since this is the main rule that causes this whole paradox. However, this would mean that there are a limited number of things you can add in a set, so mathematicians modified rule 1 as well. This was implemented by Ernst Zermelo and Abraham Fraenkel to create the Zermelo-Fraenkel set theory, which created the axiom of regularity (no sets can contain themselves), therefore evading Russell's paradox. Zermelo and Fraenkel later used this set theory alongside the axiom of choice (you can choose an element from each in a collection of sets even if it is infinite) to create a new foundation of mathematics which is still being used today.

Russell’s own solution to the paradox was called ramified type theory. This theory was proposed as an alternative mathematical foundation to set theory as well to avoid the contradictions caused by set theory (i.e., Russell’s paradox). This created a hierarchy system from the lowest level of just objects or individuals to the highest level, where sets are sets of sets of sets…of an object. In this way, sets of numbers (tier 2 set) only contain individual numbers (tier one), meaning that the set can never refer to itself, avoiding the paradox. Despite it being able to avoid the paradox, the theory was too complex in comparison to the Zermelo-Fraenkel set theory, making Zermelo’s theory more accepted.

## Similar paradoxes to Russell’s Paradox

The Barber Paradox was seen as an alternative form of Russell’s Paradox, and it was shown to show the contradictions of set theory. The paradox states that the barber is the "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself? If he does shave himself, then the barber can shave people who do not shave themselves, which is against the statement. If he doesn’t shave himself, then he doesn’t shave himself, but then in the quote it says that he should shave himself, and this once again creates a contradiction. The main counter to this paradox is that no such barber exists. The next paradox, however, cannot be solved using this logic and once again is very similar to Russell's Paradox.

Another similar paradox is the Grelling-Nelson paradox named after its creators Kurt Grelling and Leonard Nelson). Firstly, you need to understand the 2 types of adjectives which consist of:

1. Autological adjectives - adjectives that describe themselves, such as the word ‘English’.

2. Heterological adjectives - adjectives that do not describe themselves, like hyphenated, since the word does not contain a hyphen.

The paradox in this case is whether ‘heterological’ is heterological.

If it is heterological, then the word describes itself, so it is autological, not heterological. However, if it is not heterological, then the word describes itself, so it is heterological.

You can get rid of the paradox by changing the definition of heterological to all adjectives that do not describe themselves; however, the paradox can arise again with the word nonautological. In the end, Grelling supported the idea of Uuno Saarnio in 1937. That challenged the concept of a word and meaning which would avoid the Grelling-Nelson paradox.

## Conclusion

Russell's paradox shows that even the simplest ideas in mathematics, which created the foundations of mathematics, can have serious flaws and contradictions. The paradox rewrote the foundations of mathematics to be a more sophisticated system like the Zermelo-Fraenkel set theory, which is still used today. The paradox created the applied versions of itself, for instance, the Barber paradox and the Grelling-Nelson paradox.

It gave rise to new philosophical perspectives, such as intuitionism, which showed the limitations of logic and the difficulties of using self-referenced statements. It questioned whether set theory and logic are sufficient in creating a foundation of mathematics, which as a result created alternative mathematical foundations such as type theory.

Overall, Russell's Paradox helped change the foundations of mathematics to be a more rigorous. This illustrates how new areas of mathematics like set theory should be thought about in more detail to avoid potentially catastrophic consequences preventing years of mathematical work from going to waste.