Apollonian Circle Packings: Hidden Symmetries and Infinite Patterns

By Arav Goel

Introduction

Apollonian circles enjoyed perhaps their most glorious resurgence in 1936, thanks to Frederick Soddy, a Nobel laureate in chemistry, not mathematics, who became so captivated by them that he immortalised their beauty in a poem. Yes, a chemist with a Nobel Prize writing poetry about circles. In this essay, not only will you discover the intricacies of one of the most beautiful geometric problems ever, but you'll also find Soddy's poem waiting for you at the end. And who knows? By the time we're finished, you might be inspired to write your own little mathematical sonnet. But before we get to Soddy's ode to circles, let's go back to where it all started.

Apollonius' problem was first documented by René Descartes in 1643 in his correspondence with Princess Elizabeth of Bohemia, one of his pupils. In a letter to her, Descartes posed the following problem:

> *« A M. la Princesse Elizabeth, etc. Touchant le Probleme : trois cercles estant donnez, trouuer le quatriéme qui touche les trois »*

Which in English translates to - "To Princess Elizabeth, etc. Concerning the Problem: Given three circles, find the fourth that touches all three."

This problem traces back even further to Apollonius of Perga, a Greek mathematician who lived in the third century B.C. Although his contributions have been somewhat overshadowed by his predecessor Euclid, partially due to much of his work being lost, his surviving text, *Κωνικα* on Conic Sections, laid the foundation for the study of ellipses, hyperbolas, and parabolas, curves that continue to play a fundamental role in mathematics today.

Apollonius wrote a now-lost manuscript called *Tangencies*, which, according to later commentators, addressed various problems involving circles drawn to be tangent to different combinations of geometric objects: three lines, two lines and a circle, two circles and a line, or three circles. The most challenging (and famous) of these cases was the latter one: finding a circle tangent to three given circles.

The true beauty of this problem lies in how it unfolds into a never-ending sequence of nested circles - Apollonian Circle Packings - creating intricate patterns that conceal deeper and elegant connections between geometry, number theory, and group theory.

But enough anticipation. Let's dive into the mathematics and see why Soddy felt inspired to write a poem about them.

Setting the Scene: Apollonius' Tangency Problem

Consider Apollonius's problem of finding a circle tangent to three given circles for figure 1 below.

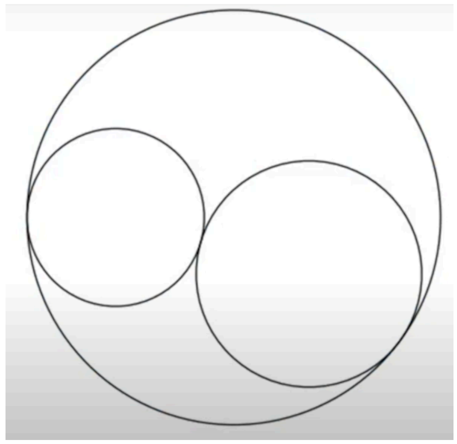

Figure 1: Apollonius' Tangency Problem - finding a circle tangent to three given circles.

You might notice that there are 2 ways:

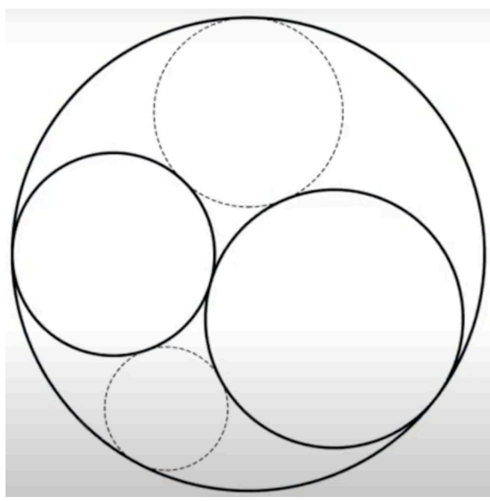

Figure 2: Two Solutions to Apollonius' Problem

Now, someone-or perhaps several mathematicians-expanded upon this problem by continuously adding new circles. As seen in Figure 2, these new circles create additional gaps where Apollonius' original question can be applied again. This process continues, producing more circles and more gaps, and then, of course, you're addicted, so you keep adding more and more circles. If you continue this process indefinitely (although you probably should not), you end up with what is known as an Apollonian Circle Packing:

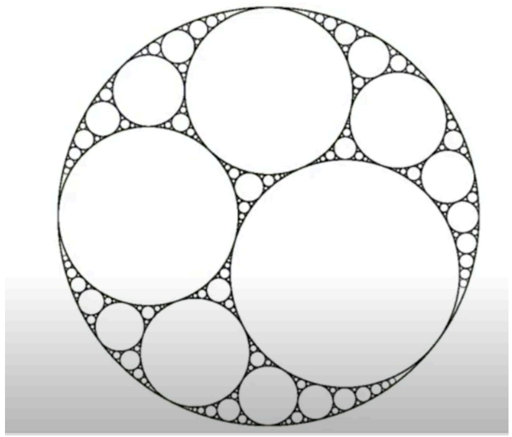

Figure 3: Apollonian Circle Packing – An arrangement of infinitely many mutually tangent circles filling the space without overlapping.

Isn't that just beautiful? A fractal made up of infinitely many circles, perfectly touching but never overlapping. The Apollonian Circle Packing is one of the earliest studied fractals to and remains a fascinating example of how complex structures can arise from simple rules.

As a fractal, the Apollonian packing has several remarkable properties:

It is a set of measure zero. To visualise this, imagine starting with a solid metal disc and drilling out an infinite series of progressively smaller circular holes (ignoring the fact that the metal is made of atoms!). You would be left with a single piece of metal, yet paradoxically, its mass would be exactly 0.

Its fractal (Hausdorff) dimension is roughly 1.30568, meaning that its complexity is less than a two-dimensional area but greater than a one-dimensional curve. When you zoom in on the Apollonian gasket, you continue to find more circles. The rate at which the number of circles increases is directly related to the Hausdorff dimension, indicating its intricate, self-repeating nature.

What I find particularly fascinating is that while it is not perfectly self-similar, portions of an Apollonian packing resemble well-known fractals. For example, if you focus on the region between three tangent circles, it is homeomorphic to the Sierpinski triangle, meaning one can be continuously transformed into the other through stretching or bending.

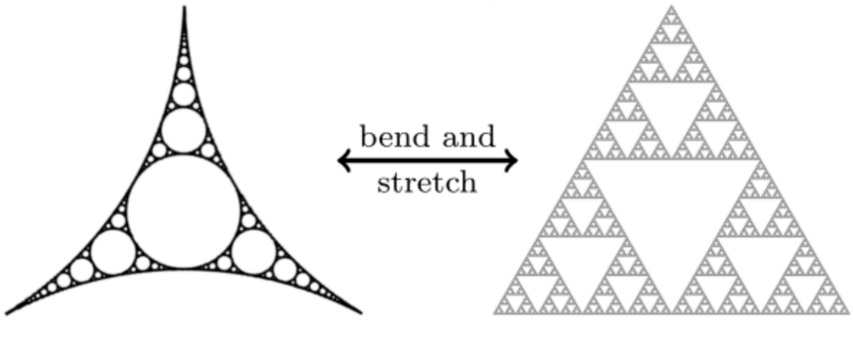

Figure 4: Apollonian Packing and the Sierpinski Triangle – A portion of an Apollonian packing is homeomorphic to the Sierpinski triangle. By compressing and stretching the packing, the circular pattern can be transformed into a triangular fractal structure.

<u>The Unexpected Link: Geometry Meets Number Theory</u>

Now it's time to delve into the inextricable link between Apollonian Circle Packings and number theory. As number theorists, we just can't resist adding some numbers to this packing; if I label every single circle with its curvature from the packing in Figure 3, where curvature is defined as the reciprocal of the radius, we notice something peculiar:

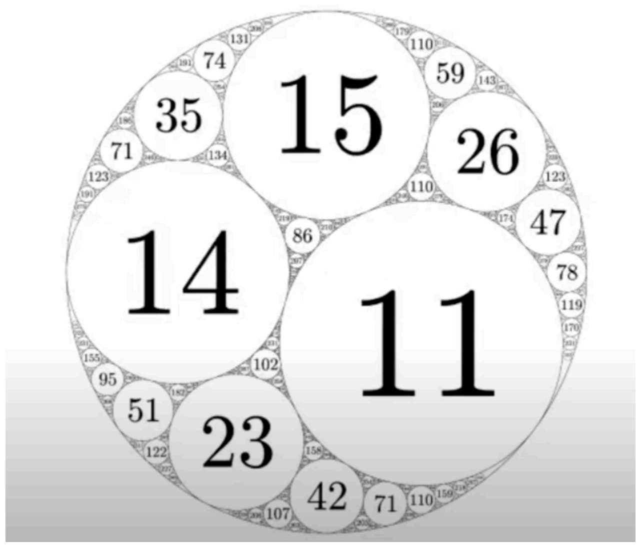

Figure 5: All the curvatures are integers! - Amazingly, in certain Apollonian Circle Packings, all the curvatures (1/radius) are integers, unveiling a deep connection between geometry and number theory.

Why is this true? After all, there's no obvious reason why these curvatures should be whole numbers. It turns out that the answer lies in some elegant geometry and algebra. To explain this, we must introduce a powerful result known as Descartes' Circle Theorem.

<u>Descartes' Circle Theorem</u>

Descartes' Circle Theorem states that for four mutually tangent circles with curvatures k_1, k_2, k_3, and k_4, the following relationship holds:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2\left(k_1^2 + k_2^2 + k_3^2 + k_4^2\right)$$

There are several different proofs of Descartes' Circle Theorem, involving those that use trigonometric identities or calculus.

Descartes and Princess Elisabeth were both familiar with Heron's formula, which they probably used in their attempts to solve our problem of mutually tangent circles. This is the method that I find most elegant (and simple). Heron's formula provides us with a way to determine the area of a triangle when all we know about a triangle is the length of its sides.
Given a triangle, its sides a, b, and c in length, Heron's formula states that:

$$\text{Area} = \sqrt{o(o-a)(o-b)(o-c)} \quad \text{with} \quad o = \frac{a+b+c}{2}.$$

Using Heron's formula, we can prove Descartes' Circle Theorem and using this, the Integral Apollonian Packing Theorem. This approach will ultimately reveal why all the curvatures in Figure 5 are integers.

<u>Proof of Descartes' Circle Theorem and the Integral Apollonian Packing Theorem</u>

The following proof involves some technical steps using Heron's formula. If you're just here for the big picture, feel free to skip ahead! But if you love a good mathematical puzzle, I encourage you to stick around. Descartes' Circle Theorem is about to reveal its secrets.

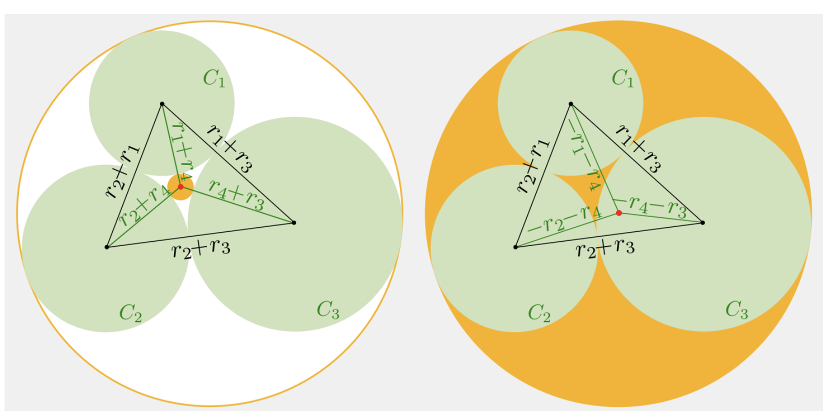

Figure 6: Three mutually touching circles C_1, C_2 and C_3 and the two solutions of Apollonius' problem side-by-side (left: a smaller circle nestled between the original 3, right: a larger circle enclosing them all).

r_1, r_2, r_3 are the radii of the three mutually tangent circles C_1, C_2 and C_3

Let's focus on the left-hand scenario from Figure 6, where the fourth circle is nestled between the other three. By connecting the centers of these four circles, we form 4 triangles. Notice that the largest triangle's area is equal to the sum of the areas of the other three smaller triangles. Using Heron's formula, we can express this relationship as:

$$\sqrt{r_1r_2r_3(r_1+r_2+r_3)} = \sqrt{r_1r_2r_4(r_1+r_2+r_4)}+\sqrt{r_1r_3r_4(r_1+r_3+r_4)}+\sqrt{r_2r_3r_4(r_2+r_3+r_4)}. \quad (1)$$

In the right-hand scenario, the fourth circle touches the three given circles externally. By convention, we assign this external circle a negative radius, which corresponds to a circle with curvature |k| that contains the other circles within its interior. Using a negative radius allows us to apply the same equation to both scenarios, whether the fourth circle is nested inside or enclosing the others. This clever trick simplifies our calculations significantly.

Let's return to the left-hand case. Solving the equation above in the traditional way by repeatedly squaring leads to some monstrous calculations. However, by carefully simplifying in each step, the final result emerges quite neatly:

To simplify our calculations and organize our proof, we will introduce several variables. These will help us manipulate the equation more effectively. Let us define:

$$\begin{aligned} s &= r_1+r_2+r_3+r_4, \quad \text{(the sum of the four radii)} \\ p &= r_1r_2r_3r_4, \quad \text{(the product of the four radii)} \\ t &= \frac{p}{s}, \quad \text{(a ratio of the product to the sum)} \\ u &= \frac{1}{s}. \quad \text{(the reciprocal of the sum)} \end{aligned}$$

Furthermore, we define:

$$\begin{aligned} \alpha &= \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}, \quad \text{(the sum of reciprocals of the radii)} \\ \beta &= \frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}. \quad \text{(the sum of squared reciprocals)} \end{aligned}$$

Using these definitions, we can rewrite equation our original expression as:

$$\sqrt{r_1r_2r_3s-p} = \sqrt{r_1r_2r_4s-p}+\sqrt{r_1r_3r_4s-p}+\sqrt{r_2r_3r_4s-p}.$$

We now divide by $\sqrt{s}$ and rearrange:

$$\sqrt{r_1 r_2 r_3 - t} - \sqrt{r_1 r_2 r_4 - t} = \sqrt{r_1 r_3 r_4 - t} + \sqrt{r_2 r_3 r_4 - t}.$$

Squaring both sides and rearranging leads to:

$$r_1 r_2 r_3 + r_1 r_2 r_4 - r_1 r_3 r_4 - r_2 r_3 r_4 = 2\left(\sqrt{r_1 r_2 r_3 - t}\sqrt{r_1 r_2 r_4 - t} + \sqrt{r_1 r_3 r_4 - t}\sqrt{r_2 r_3 r_4 - t}\right).$$

We divide this result by p:

$$\frac{1}{r_4} + \frac{1}{r_3} - \frac{1}{r_2} - \frac{1}{r_1} = 2\left(\sqrt{\frac{1}{r_4} - u}\sqrt{\frac{1}{r_3} - u} + \sqrt{\frac{1}{r_2} - u}\sqrt{\frac{1}{r_1} - u}\right).$$

Again, we square both sides:

$$\beta + \frac{2}{r_3 r_4} - \frac{2}{r_2 r_4} - \frac{2}{r_1 r_4} - \frac{2}{r_2 r_3} - \frac{2}{r_1 r_3} + \frac{2}{r_1 r_2} = \frac{4}{r_3 r_4} + \frac{4}{r_1 r_2} - 4\alpha u + 8u^2 + 8\sqrt{\frac{1}{r_4} - u}\sqrt{\frac{1}{r_3} - u}\sqrt{\frac{1}{r_2} - u}\sqrt{\frac{1}{r_1} - u}.$$

or, after rearranging:

$$\beta - \frac{2}{r_3 r_4} - \frac{2}{r_2 r_4} - \frac{2}{r_1 r_4} - \frac{2}{r_2 r_3} - \frac{2}{r_1 r_3} - \frac{2}{r_1 r_2} + 4\alpha u - 8u^2 = 8\sqrt{\frac{1}{r_4} - u}\sqrt{\frac{1}{r_3} - u}\sqrt{\frac{1}{r_2} - u}\sqrt{\frac{1}{r_1} - u}.$$

Note: Since

$$\alpha^2 - \beta = \frac{2}{r_3 r_4} + \frac{2}{r_2 r_4} + \frac{2}{r_1 r_4} + \frac{2}{r_2 r_3} + \frac{2}{r_1 r_3} + \frac{2}{r_1 r_2}$$

We can rewrite this equation as:

$$(2\beta - \alpha^2) + 4\alpha u - 8u^2 = 8\sqrt{\frac{1}{r_4} - u} \cdot \sqrt{\frac{1}{r_3} - u} \cdot \sqrt{\frac{1}{r_2} - u} \cdot \sqrt{\frac{1}{r_1} - u}.$$

Squaring both sides (again), we find for the left-hand side:

$$(2\beta - \alpha^2)^2 + 8(2\beta - \alpha^2)\alpha u - 16(2\beta - \alpha^2)u^2 + 16\alpha^2 u^2 - 64\alpha u^3 + 64u^4.$$

The right-hand side is given by:

$$\frac{64}{r_1r_2r_3r_4} - 64\left(\frac{1}{r_1r_2r_3} + \frac{1}{r_1r_2r_4} + \frac{1}{r_1r_3r_4} + \frac{1}{r_2r_3r_4}\right)u + 32(\alpha^2 - \beta)u^2 - 64\alpha u^3 + 64u^4.$$

The first two terms on the right-hand side cancel out, and a number of terms of left and right-hand side are equal. Hence we get after rearranging:

$$(2\beta - \alpha^2)^2 + 8(2\beta - \alpha^2)\alpha u = 0 \quad \text{or} \quad (2\beta - \alpha^2) \cdot (2\beta - \alpha^2 + 8\alpha u) = 0.$$

The second factor cannot be zero, since in that case $2\beta - \alpha^2 = -8\alpha u$ which results in a negative left-hand side from earlier. Hence, we have that:

$$2\beta = \alpha^2, \quad \text{or} \quad 2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right) = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2.$$

□

Note: Given r_1, r_2, r_3, this is a quadratic equation for r_4 with two solutions: the radii of the two tangent circles in Figure 6

We’ve just done the enormous task of proving Descartes’ Circle Theorem! It was a rigorous journey, but now we can enjoy the benefits of this newfound theorem.

So, what’s the big deal? Why did we work through all that algebra? Well, as promised, it turns out this theorem is the crucial tool we needed to prove the Integral Apollonian Packing Theorem (as illustrated in Figure 5, where all labeled curvatures are integers).

The Integral Apollonian Packing Theorem states that if the initial four circles in a packing have integer curvatures, then every other circle added through the packing process will also have integer curvature.

But, how does Descartes' Theorem help us prove this: given three mutually tangent circles of curvatures a, b, and c, there are exactly two circles that are tangent to all three. Let’s call their curvatures d and d'. Descartes’ Circle Theorem tells us that:

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$$

$$2(a^2 + b^2 + c^2 + (d')^2) = (a + b + c + d')^2$$

Subtracting the second equation from the first, we get:

$$d + d' = 2(a + b + c)$$

So if a, b, c and d are integers, it follows that d′ is also an integer. This is because the sum of integers is always an integer, and Descartes' theorem makes sure that this property persists as the packing progresses. Therefore, if the initial four circles in an Apollonian packing have integer curvatures, all subsequent circles will also have integer curvatures.

I find that simply stunning: at first glance, Apollonian Circle Packings seem chaotic: a mess of circles crammed into every available gap, getting smaller and smaller endlessly. Yet, beneath this apparent random arrangement lies such a beautiful structure determined by integer curvatures. It's the perfect example of harmony between discrete and continuous mathematics.

<u>Local-Global Conjecture and Modulo Restrictions</u>

Now, let's take this a step further. We've established that the curvatures in an Apollonian Circle Packing can be integers. But here's an even more fascinating (and harder) question: *which* integers do you get?

When mathematicians started examining these curvatures, they noticed something curious. Let's colour each circle from Figure 5 by the remainder when dividing its curvature by 3 (or considering all numbers mod 3):

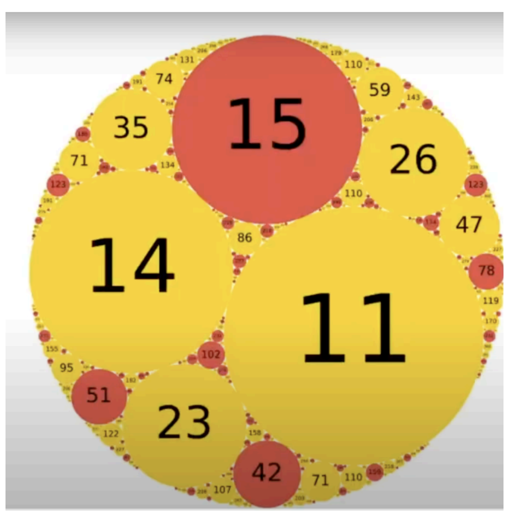

Figure 7: Modulo 3 Colouring of Integral Apollonian Packing Curvatures - Disks are coloured by their curvature modulo 3. Red disks correspond to curvatures that are 0 mod 3, and yellow disks correspond to curvatures 2 mod 3.

But where are the disks labeled '1 mod 3'? They're nowhere to be found. Intrigued, you decide to repeat the experiment with modulo 5...

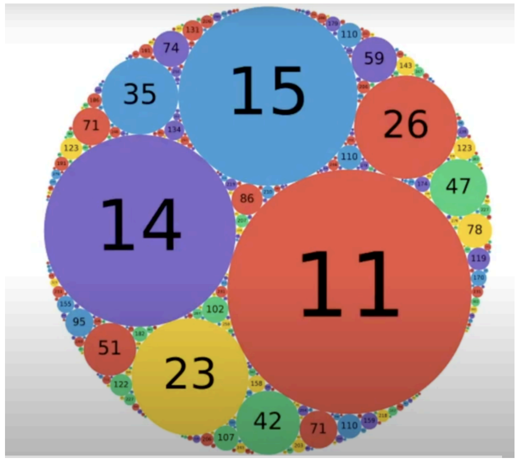

Figure 8: Modulo 5 Colouring of Apollonian Packing Curvatures - Disks are coloured according to their curvature modulo 5. Each colour represents a different residue class modulo 5. Unlike the modulo 3 case, all five residue classes appear, suggesting a more complete distribution.

And now suddenly, all five residue classes are present. What makes 3 so special? And could this phenomenon happen with other numbers too?

If we keep experimenting, we might come up with some interesting observations:

Observation 1: The modulo restriction can only happen modulo 24. That is, some remainders modulo 24 are allowed, while others simply aren't. Notice how 3 divides 24, but 5 doesn't.

Observation 2: Not all positive integers appear as curvatures, but many do**.** In this example, the outer curvature is 6, or really −6, because it contains the circles rather than being contained within them. But after that, there's another gap until 11. However, beyond these gaps, something remarkable happens - it *looks* like every positive integer eventually shows up, provided it belongs to one of the allowed residue classes modulo 24. This observation is called the Local-Global Conjecture for Apollonian Circle Packings.

Proving Observation 1: While proving observation 1 is possible, it's beyond the scope of this essay. It involves a beautiful use of quadratic residues, inherent properties of Descartes' Circle Theorem and the use of Apollonian group actions (as shall be introduced shortly). If you're curious, I highly recommend reading https://www.math.ucdavis.edu/~efuchs/efuchsthesis.pdf

Proving Observation 2: Here's the twist, the local-global conjecture was widely accepted, with substantial numerical evidence to back it up. However, in 2023, a team from the University of Colorado Boulder discovered counterexamples that proved the conjecture false! Far from being a disaster, this discovery has paved the way for many new avenues of research. Understanding why certain numbers are missing and developing methods to predict these gaps has become one of the most intriguing and challenging problems in the field.

The secret weapon for tackling these problems that number theorists use to explore these observations is the Apollonian group. This group tells us how curvatures evolve within the packing by giving us a rigorous framework to study their arithmetic properties.

Group Theory and Apollonian Packings

Apollonian group

We think of the group as something that acts on quadruples of circles. Just like the moves of a Rubik's Cube change its colours around, the moves here change which 4 mutually tangent circles you're focusing on, within the whole packing. And just like twisting a Rubik's Cube, there are four distinct "moves" you can make. Each move is about swapping out one circle for its alternate solution to Apollonius' problem:

The diagram below shows a particular quadruple of mutually tangential circles 4 times, labelled S_1, S_2, S_3 and S_4.

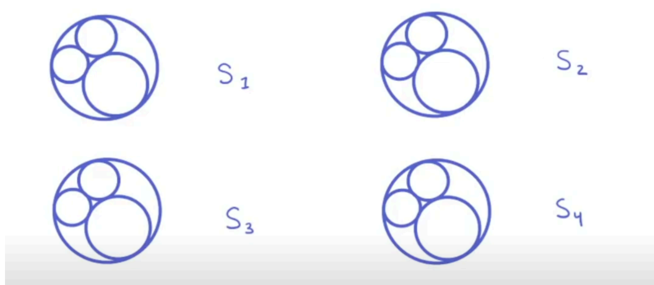

Figure 9: The Initial Configuration of an Apollonian quadruple.

To make a move, you select one circle from the quadruple (highlighted in red):

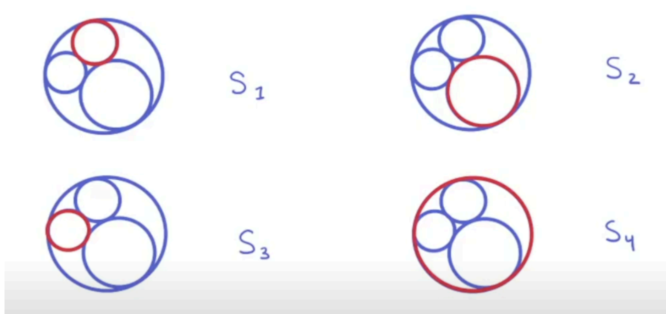

Figure 10 -The Four Generating Moves of the Apollonian Group

The trick is to imagine the other three circles as a new Apollonius problem and replace the selected circle with its alternate solution to that problem. This operation generates a new packing configuration:

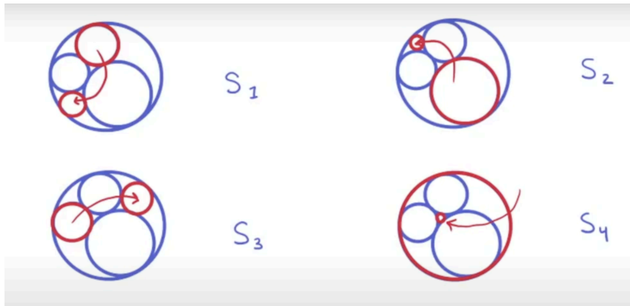

Figure 11 - Alternate Solutions Generated by Apollonian Group Moves

Excitingly, just these 4 operations shift us through the entire packing configuration, effectively allowing us to explore all possible circle curvatures within the Apollonian packing.

Let us put these Apollonian Group Moves into action with an example. Here I've highlighted a sort of base quadruple in blue (a starting point - corresponding to the identity element in groups):

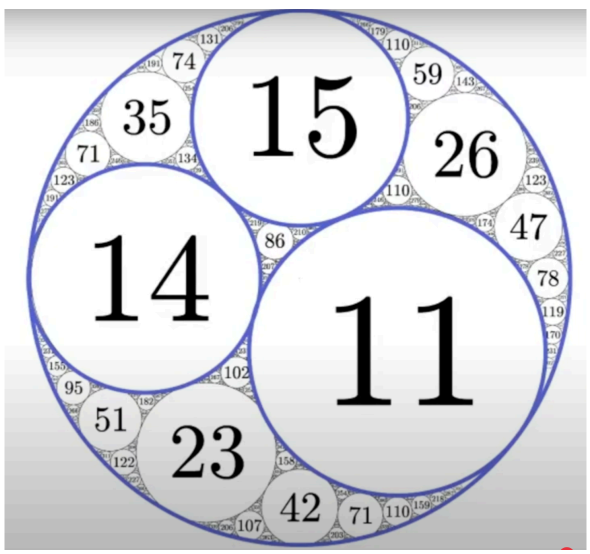

Figure 12: The Base Quadruple - Your Starting Point in the Apollonian Group

I can also now keep track of the 4 curvatures on which I perform my group action as a 4-dimensional vector.

$$\begin{pmatrix} -6 \\ 11 \\ 15 \\ 14 \end{pmatrix}$$

Now, let's see how applying group moves transforms this vector. For example, if I apply S_2, I move to a new quadruple by swapping out the 11 circle with a 35. If I then follow up with an S_3, I swap the 15 with a 71, then S_4 replaces the 14 with 186. So by applying elements of the group, we move around the packing from quadruple to quadruple.

$$I \to S_2 \to S_3 S_2 \to S_4 S_3 S_2$$

$$\begin{pmatrix} -6 \\ 11 \\ 15 \\ 14 \end{pmatrix} \to \begin{pmatrix} -6 \\ 35 \\ 15 \\ 14 \end{pmatrix} \to \begin{pmatrix} -6 \\ 35 \\ 71 \\ 14 \end{pmatrix} \to \begin{pmatrix} -6 \\ 35 \\ 71 \\ 186 \end{pmatrix}$$

Now, here's where the real magic begins. The group's action on the vector representing curvatures is, in fact, a matrix multiplication! Each group move S_i corresponds to a matrix that transforms our 4-dimensional vector of curvatures. What this means is that I can write each group element as a matrix! And then to see what happens to curvatures, I just multiply my vector of curvature by these matrices.

I'm sure you're itching to know what these matrices look like and, more importantly, how we find them. So, let's try and derive them together.

Recall from Descartes' Circle Theorem, we have

$$d + d' = 2(a + b + c)$$

Now, let's rearrange to solve for d′:

$$d' = 2a + 2b + 2c - d$$

Can you spot one of the matrices now? The equation above is telling us that when we swap one curvature d for another d', the new curvature is just a linear combination of the original curvatures. And that's the clue we needed:

Matrix form:

$$\begin{pmatrix} a \\ b \\ c \\ d' \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

This matrix tells us exactly how to swap out d for d' while maintaining the curvatures' underlying structure. And the best part? There's a matrix like this for each of the four moves S_1 ,S_2,S_3 and S_4:

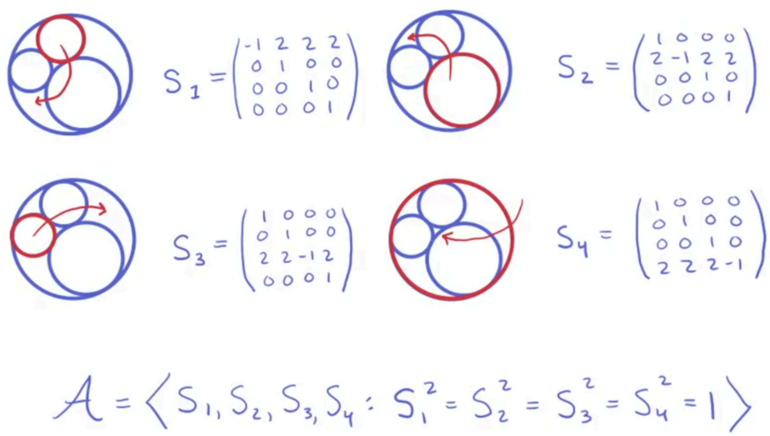

Figure 13: Matrix Representations of the Four Generating Moves

The group is generated by these 4 elements, and they satisfy the crucial relationship that each one is an involution, meaning if you perform the move twice, it undoes itself, so $S^2_i = 1$, for i = 1,2,3,4.

So, how does this group help us answer our initial question about which type of curvatures appear within an integral Apollonian packing? We can now exploit the symmetries of groups to reveal which curvatures are accessible. One of the most powerful techniques in visual group theory for uncovering these symmetries is to draw the Cayley Graph of the group. A Cayley Graph is a visual representation of a group where each node represents a particular configuration (in our case, a quadruple of curvatures) and each edge corresponds to the application of a group generator S_i and because each element is an involution ($Si^2=1$), moving twice along the same edge brings you back to where you started.

The beauty of these diagrams is that it lets us *see* how all the curvatures are related by the group's moves

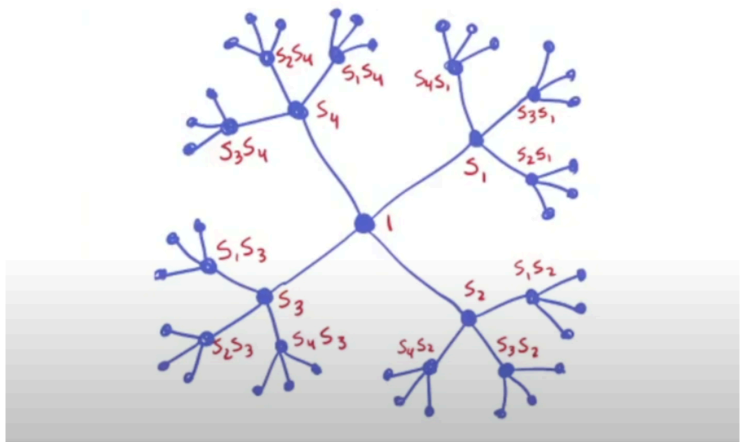

Figure 14: Cayley Graph of the Apollonian Group - This graph shows the structure of the Apollonian group, with each node representing a particular configuration of curvatures and each edge relates to the use of a generator.

Start with the identity at the centre and draw an edge for each generator S_i. Travelling along that edge is the equivalent to multiplying by that element in the group.

To make the graph even more insightful, I could also decorate the graph with corresponding quadruples of curvatures:

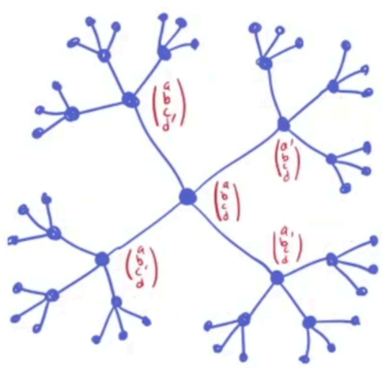

Figure 15: Cayley Graph Decorated with Quadruples of Curvatures

So if I pick one to put at the origin, then I can check how the group acts on it, along each edge. For example, as I travel along an S_1 edge, how does it change the quadruples? Then all the labels I get are called the **orbit** of the group. By following these paths, we can observe how different curvature values are generated.

This graph is a critical tool in understanding why certain curvatures are missing under specific modulo conditions (i.e. extensions of the local-global conjecture). Now here's the trick. I take my original Cayley graph representing the orbit of the Apollonian group and reduce all the quadruple vectors mod p, where p is a prime:

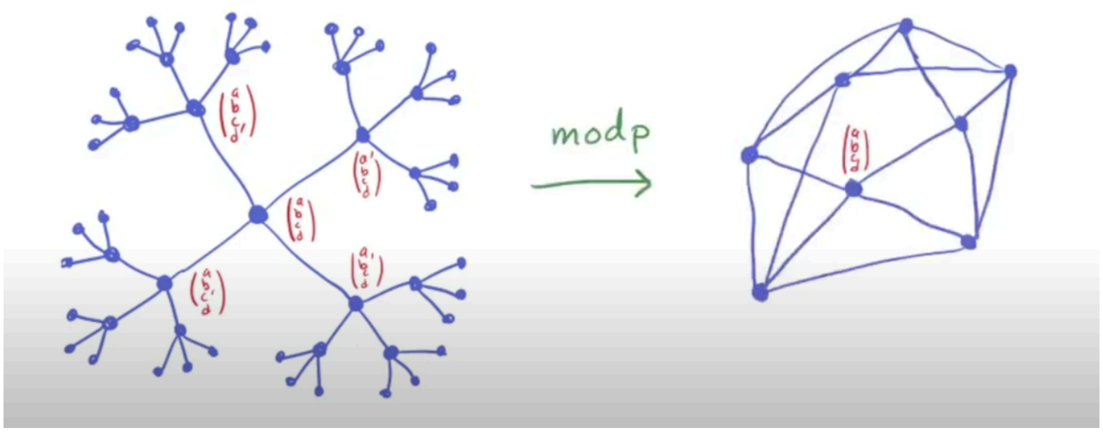

Figure 16: Reduced Cayley Graph Modulo p - Nodes that are congruent modulo p are identified, creating a simplified, finite graph that reveals which curvatures are achievable under modulo conditions.

This process collapses the infinite graph into a finite structure by identifying vertices that are congruent modulo p. By studying these smaller graphs, and seeing how the bigger graph covers the smaller graph, I can argue that certain remainders modulo p can occur and others can't, and as you travel outward through the Apollonian group, you tend to get all the allowed remainders regularly and randomly.

Let's take a moment to recap what we've uncovered in this essay.

Recap

Wow, what a journey it has been! We started with Descartes' Circle Theorem, proving it using Heron's Formula and showing what this meant for Integral Apollonian Packings. Then, we explored the fascinating world of curvatures, discovering an intricate structure determined by the Apollonian group.

Along the way, we encountered group theory, developing an understanding of how the group acts on quadruples of curvatures. We represented this action through matrix multiplication, which allowed us to explore the possible configurations of circles. By drawing the Cayley Graph, we were able to make these configurations visual and examine their structure.

But we didn't decide to stop there. Finally, by reducing the Cayley Graph modulo p, we unveiled an entirely new perspective. Suddenly, what was once an infinite graph becomes something finite and manageable. And through these reductions, we can discover which curvatures can appear and which are forbidden modulo p - exciting clues towards understanding deeper mysteries of the Apollonian packing.

And yet, we're still left with burning questions. The Local-Global Conjecture was thought to provide a complete description, but we now know there are unexpected gaps. What's causing these gaps? Can we classify them all? The journey is far from over, and the mysteries continue to unravel.

The pen is now in your hand. Armed with the tools we've covered today: Descartes' Circle Theorem, matrix actions of the Apollonian group, Cayley graphs, and modular reductions, you are ready to delve deeper and explore new paths in this mathematical landscape.

And now, the moment you've all been waiting for. Remember that poem I promised at the start? This was published by Nature in 1936, by Frederick Soddy (the Nobel Laureate in Chemistry), enthralled by Descartes' theorem.

He called the poem "The Kiss Precise"

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

The first stanza simply describes Apollonius' problem, referring to the curvature as 'bend'. Now that we have uncovered Descartes' Circle Theorem, hopefully, the meaning of the final stanza should click into place. Its elegant rhyme captures the very formula we derived the very formula we derived:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2\left(k_1^2 + k_2^2 + k_3^2 + k_4^2\right)$$

The poem captures the essence of the theorem with such precision that it feels poetic in its own right. And perhaps that's the true beauty of mathematics, when complexity and elegance meet so seamlessly that such a simple verse can cover an entire field of ideas.

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