Turtles All The Way Down

The Thue-Morse Sequence and Doodling in Math Class

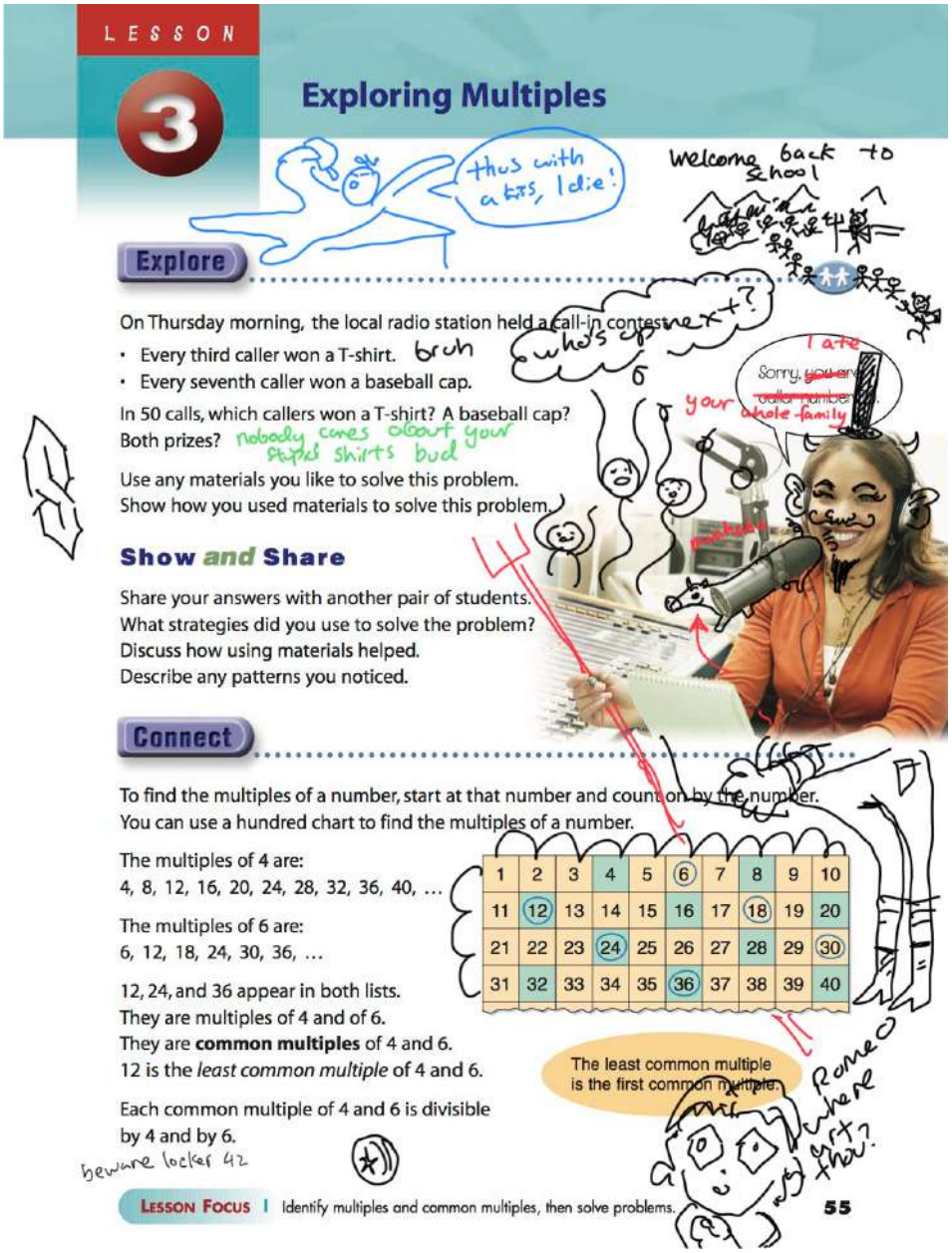
By River Phoenix

tur·tle /ˈtərd(ə)l/ : A symbol of infinite regress. An idea that you live life on top of a turtle, which itself lives on a bigger turtle, and below that turtle an even bigger turtle, on and on to infinity. I wonder where all these turtles came from.

So you're distracted in math class and scribbling lines in your school's expensive textbook while your teacher drones on and on about stupid equations and symbols or whatever. Wait, what was she talking about again? It's not made any better than by the fact that she's also the gym teacher substituting today and hasn't touched this stuff for years.

Anyways, you continue defacing school property but you get tired of drawing beards and mustaches on people, partially because it seems like you weren't the first person to decide that this was the best way to spend the prime of your limited time on this earth where every decision you make will be crucial in determining your future and have consequences for the rest of your li- wait, is this a permanent marker?

Oh well, too late now.



LESSON 3

Exploring Multiples

thus with a kiss, I die!

welcome back to school

Explore

On Thursday morning, the local radio station held a call-in contest.

- Every third caller won a T-shirt.
- Every seventh caller won a baseball cap.

In 50 calls, which callers won a T-shirt? A baseball cap? Both prizes?

Use any materials you like to solve this problem.
Show how you used materials to solve this problem.

Show and Share

Share your answers with another pair of students.
What strategies did you use to solve the problem?
Discuss how using materials helped.
Describe any patterns you noticed.

Connect

To find the multiples of a number, start at that number and count on by the number.
You can use a hundred chart to find the multiples of a number.

The multiples of 4 are:
4, 8, 12, 16, 20, 24, 28, 32, 36, 40, ...

The multiples of 6 are:
6, 12, 18, 24, 30, 36, ...

12, 24, and 36 appear in both lists.
They are multiples of 4 and of 6.
They are **common multiples** of 4 and 6.
12 is the *least common multiple* of 4 and 6.

Each common multiple of 4 and 6 is divisible by 4 and by 6.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40

The least common multiple is the first common multiple.

LESSON FOCUS | Identify multiples and common multiples, then solve problems.

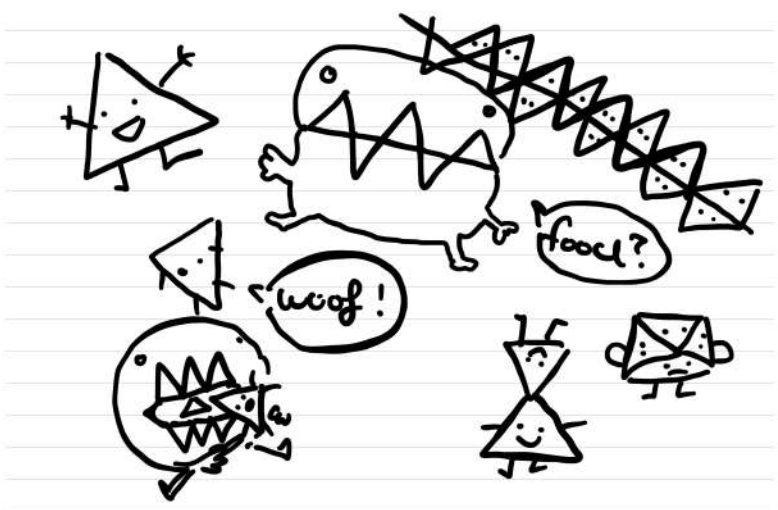
55

You decide to switch it up a bit and instead of squiggly lines, you start making sharp zigzags. Whatever this is, it's definitely better than math class.

The Zigzags start getting repetitive, but you realize that if you connect them you get even cooler shapes and you see that your zigzags are now turning into triangles.

I guess you have a pretty short attention span, which is one of the reasons you haven't been doing so well in school, but the whole triangle thing starts to look pretty cool.

Triangles Here



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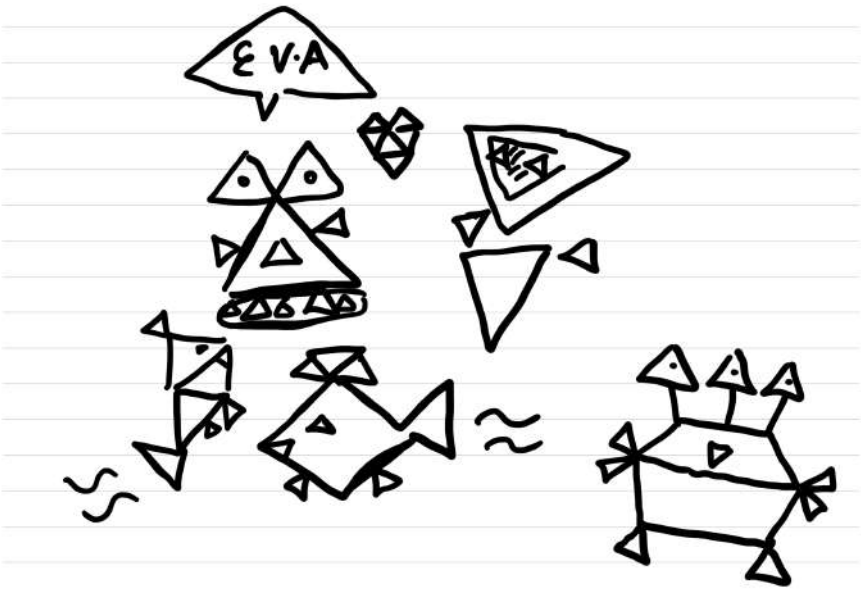
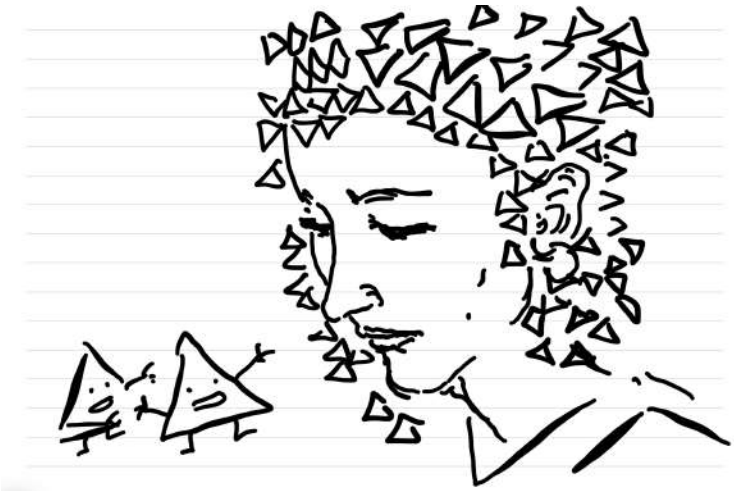


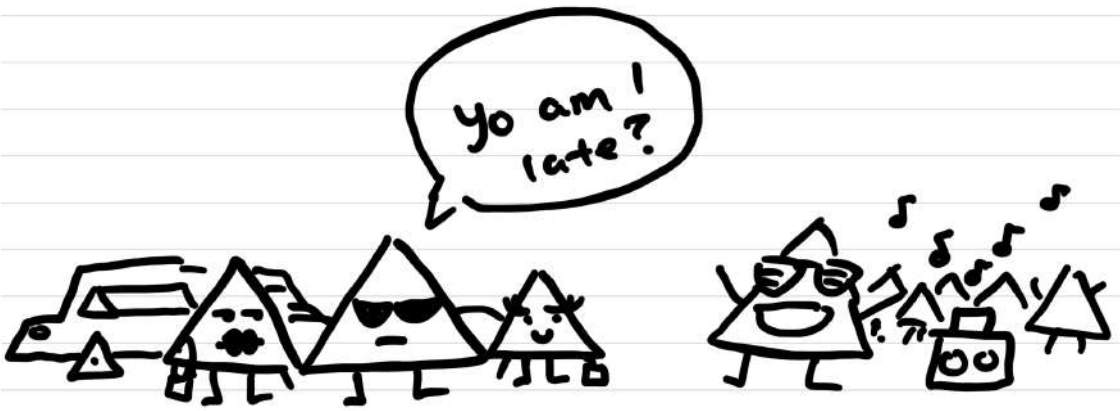
Triangles There



Triangle Triangle Triangle

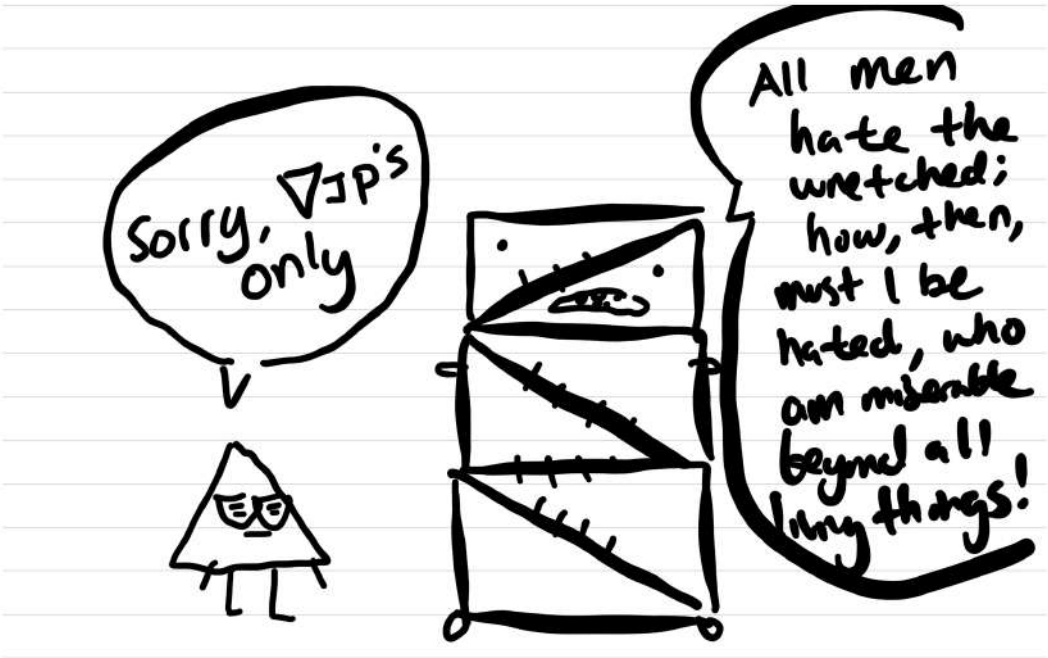
You come to the revelation that even though one squiggle, one zigzag, and one triangle may be interesting and unique, it's a lot more fun and interesting to draw them with each other.

After that revelation, you try to find new ways for the triangles to get together to mix and mingle.

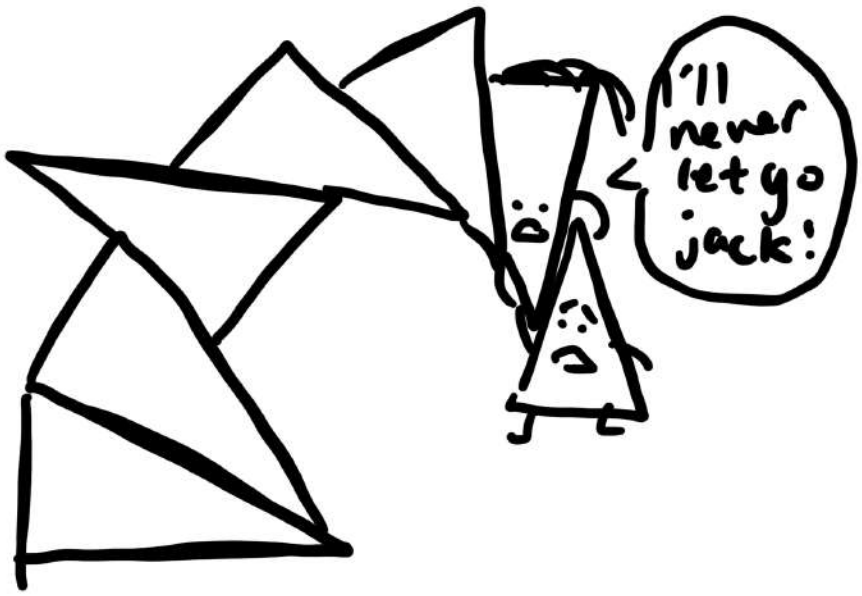


You already tried to balance one triangle on top of another so now it's a challenge to balance as many triangles on top of each as possible.

You want to do this while also making sure that the triangles don’t stack directly on top of each other since then you’re basically stacking rectangles.

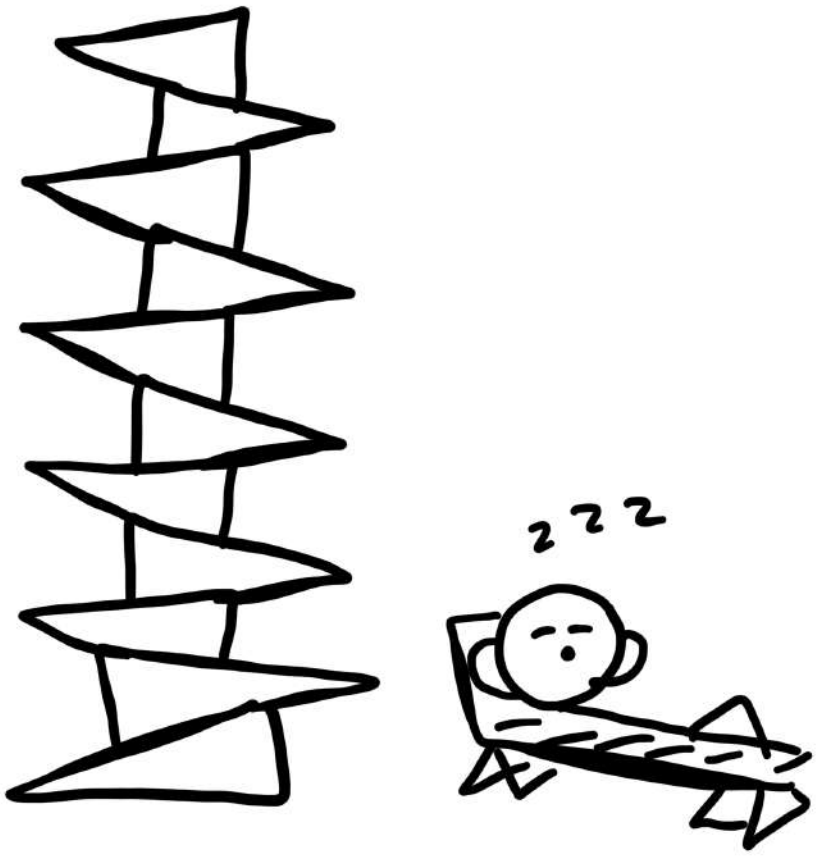


You also want to make sure the triangle party can go as high as possible while not becoming too heavy on one side and falling over. You're running out of room so you draw using the extra space on your desk.



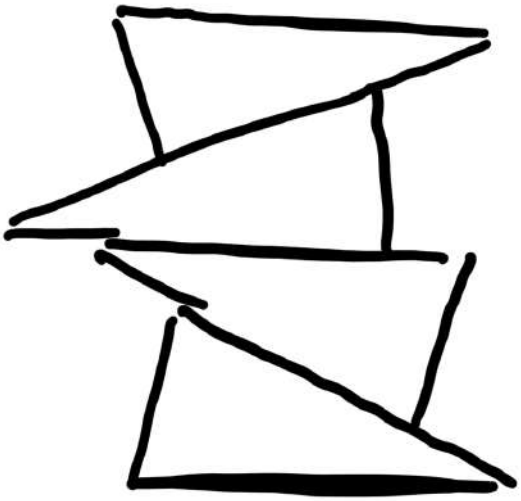
You see that if you stack all the triangles together facing either direction, they eventually curve in on themselves.

You try again but this time you alternate the triangles each time. The third time's the charm, but stacked this way the triangles look pretty boring and the pattern's easily predictable.



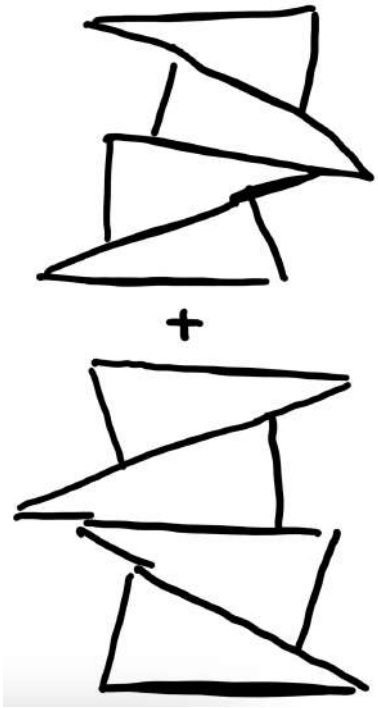
Your OCD can't let this go, so you crumble up your paper and start over.

You begin now with the triangles alternating, but this time to avoid repetition you repeat the second triangle's direction, however, if you keep going the triangle tower's gonna topple over, so you end with the fourth triangle pointing the opposite way.

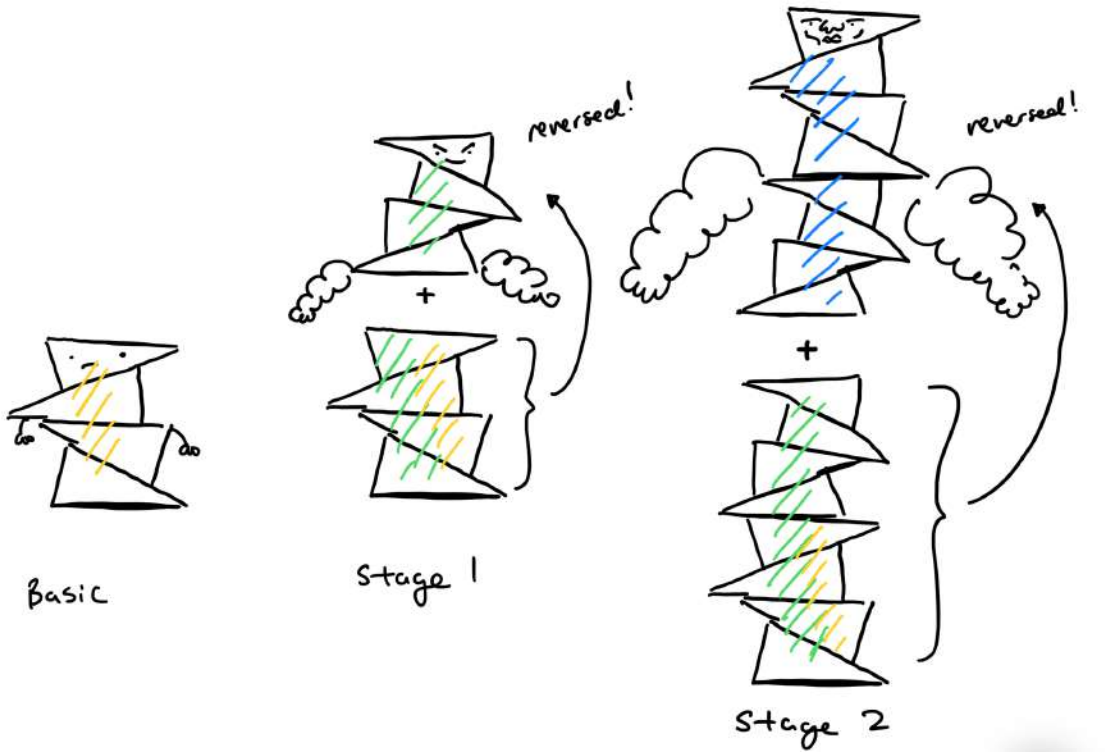


The 4 triangle pattern this time is pretty interesting but if you keep repeating it it'll just be the alternating predictable pattern all over again but with extra steps.

You think about it for a while and come up with an obvious way to keep the tower balanced while keeping the triangle pattern funky. Since the pattern you have right now is already balanced, you would just create a new stack by taking what you have and flipping everything. All the right-pointing triangles now point left, and all the left-pointing triangles now point right. Then you stack one on top of the other.



You keep going for a while and eventually, you figure out that if you just take the number of all the triangles that you've already drawn, flip their directions and stack them on top then you can have an endless

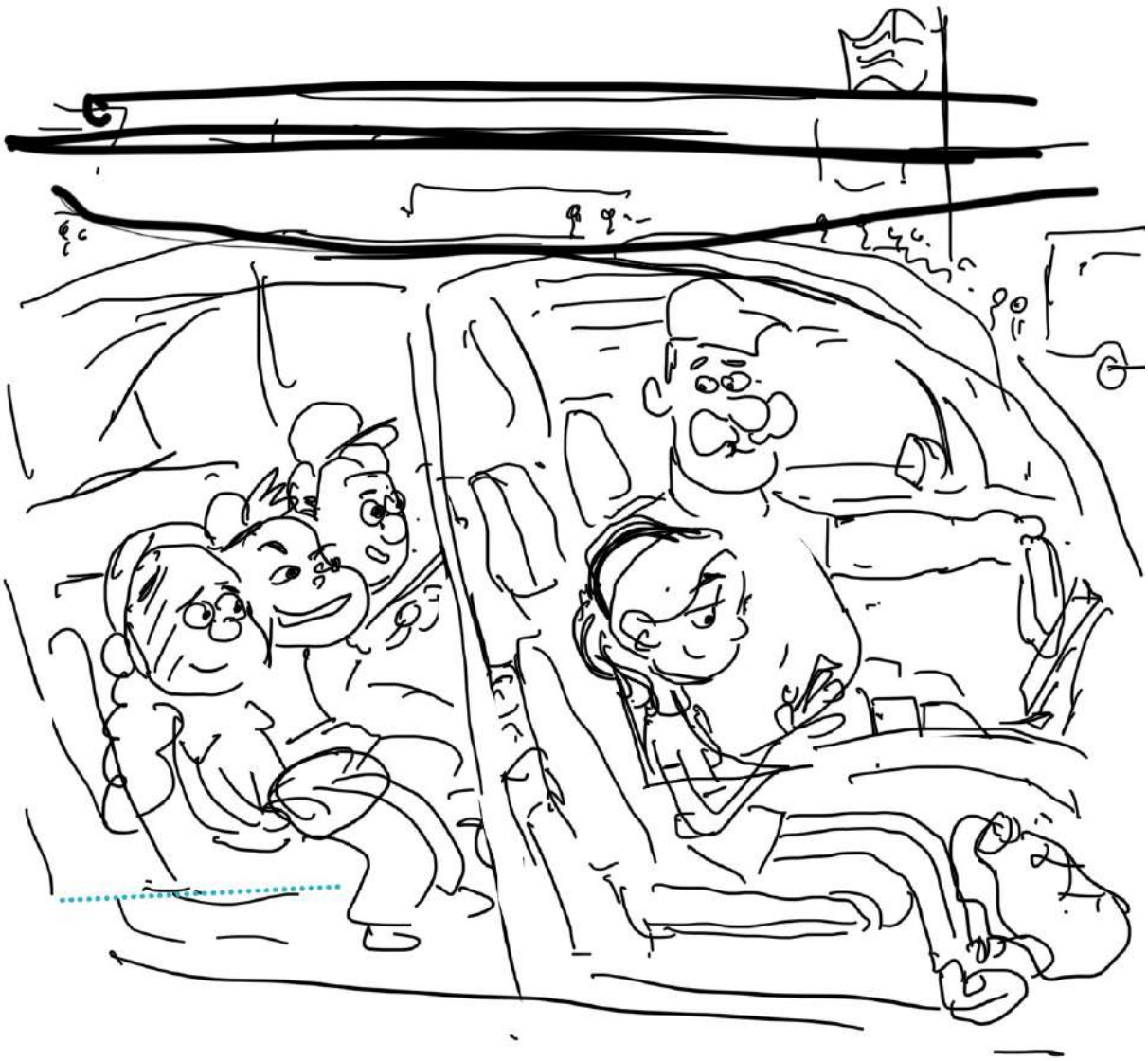


balanced tower of triangles without having a repetitive pattern.

This is cool, way better than math class.

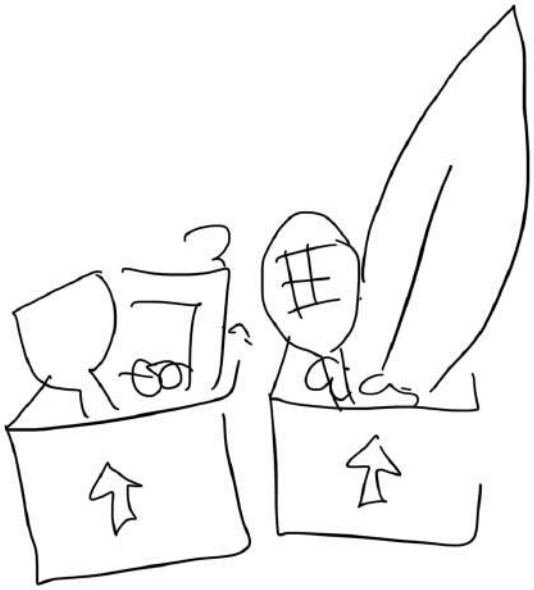
You look up from your desk just in time to see kids secretly sneaking out the door right before the school bell suddenly rings. The papers get shoved down the bag, and the teacher's last-minute reminders get drowned out by the roar of kids gushing through the halls, outside engines humming, flags a-flapping, the smell of the world rushing back in with the breeze.

You and your friends get in your dad’s car. Since you all row junior crew, he’s taking you guys to the school's rowing intramurals this afternoon. Your sister’s in there too, she’s ignoring you to talk to her boyfriend. Dad’s also the coach.



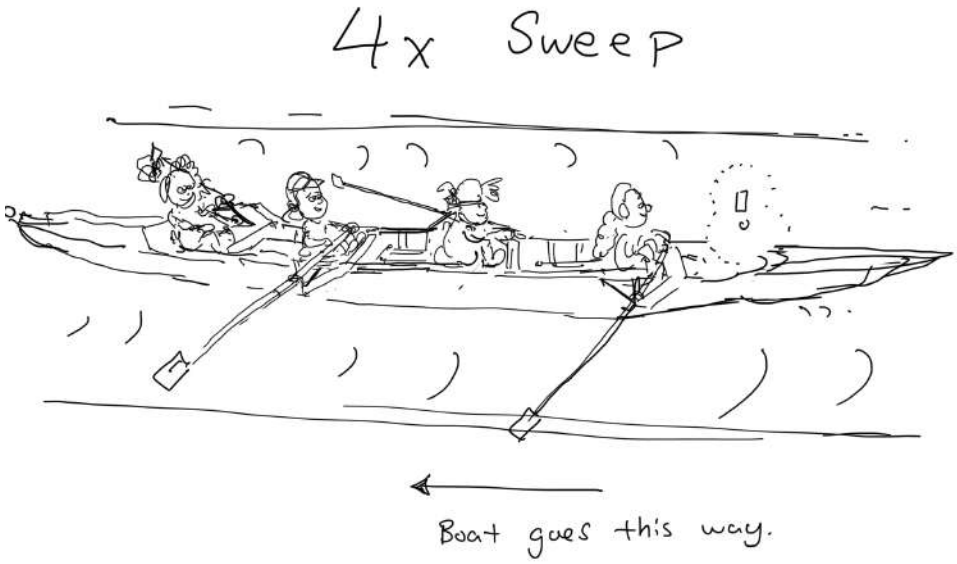
Before getting in the car, you guys have to move some boxes to the back to fit. The boxes are from Mom’s house. Your parents are getting divorced and your Dad’s taking stuff to his motel. They haven’t agreed

on who gets what yet even though they've been at it for a few weeks. No wonder he doesn't seem to be in a good mood today.



Since the property is being split up, the divorce has come down to a first-come-first-serve basis, your dad's taken the only car and so has the responsibility to drive your sister to her dance recitals.

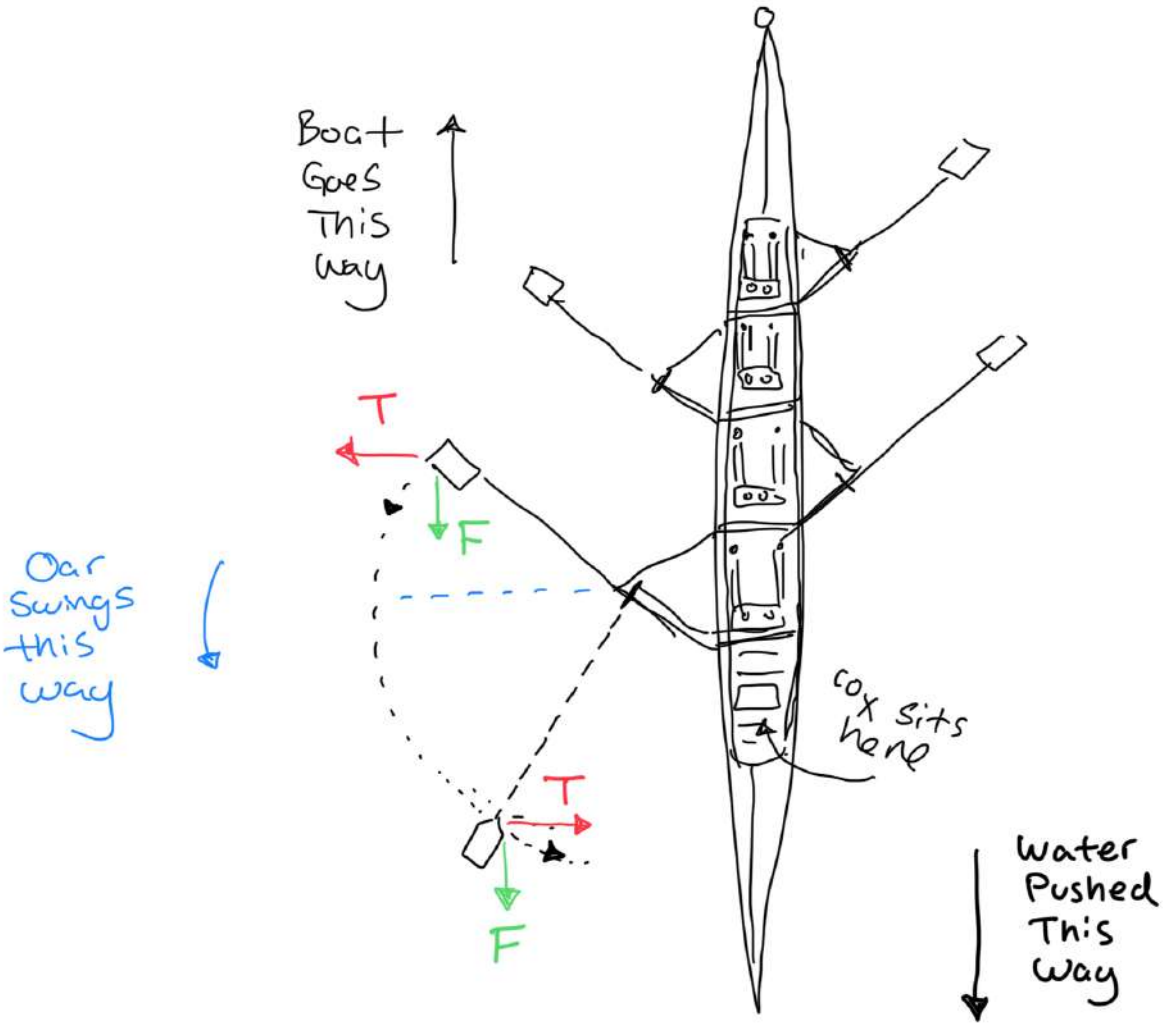
Anyway, you and your friends finally get dropped off. You wave goodbye to your Dad, and you guys grab your stuff and head down to the pier to meet up with everyone else.



You all row a “4x+” which is a super narrow boat that holds 4 rowers holding one oar each. Your rowboat is also called a “shell”.

You also have a “coxswain” in the back of the shell to make sure your team don’t row too far left or too far right. This job’s probably the most important position on the boat since all the rowers are facing with their backs to where the shell is going, and only the coxswain sees forward.

To move your boat forward every rower on your team has to swing their oars periodically. When the oar is pushing the shell it traces a circular arc in the water. One arc is a single stroke.

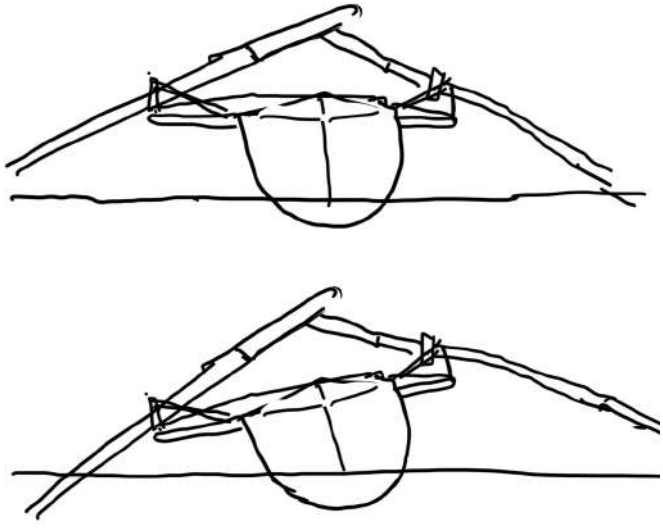


You've never learned physics before but from your intuitive understanding, you know two forces are responsible for your oar moving when you swing it. One is the force that propels the shell (F) and the other is the sideways force, the transverse force (T).

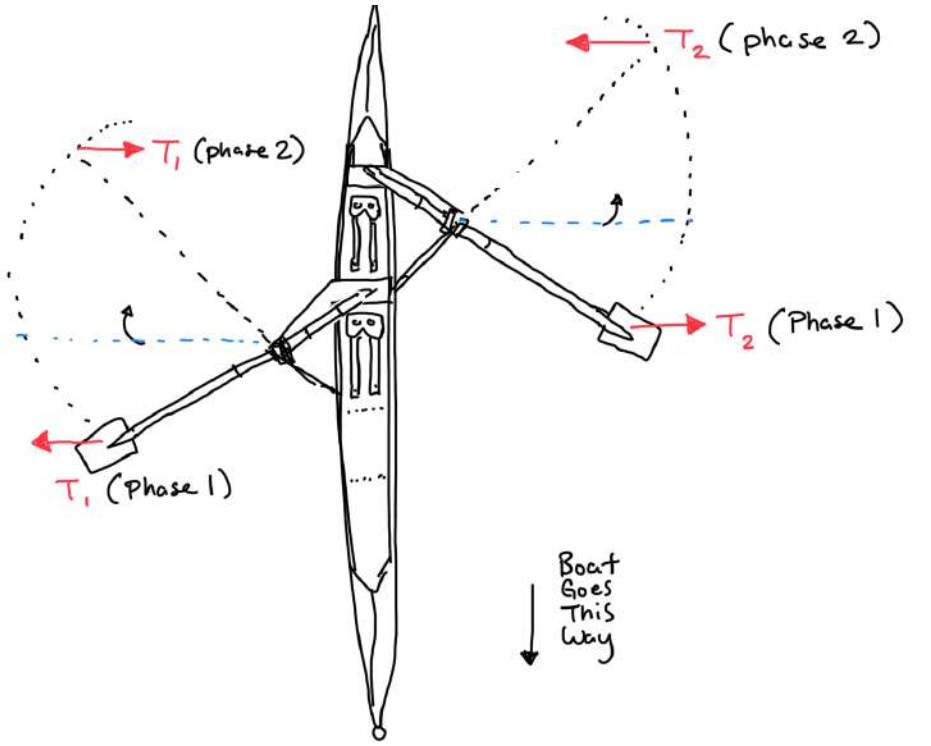
By combining the direction and force of the propelling force and the transverse force, you get the motion traced by the oar swing.

This is also why you see the transverse force changing directions in the middle of the swing. If the transverse force didn't change directions you wouldn't be able to fully swing the oar back.

Because each rower only controls one oar which pushes one side of the shell, it can be easy for one rower to swing their oar a little harder than the rower controlling the other side, or for the two side's timing to be off by a few seconds.



This creates a big problem that rowers face called the "wiggle". Due to the changing direction of the transverse force, rowers pushing their oars not in tune cause the shell to swing side to side.



(Phase 1) + (Phase 2) = One Stroke

If T_1 (Phase 1) > T_2 (Phase 1)

$\Rightarrow$ T_1 (Phase 2) > T_2 (Phase 2)

$\Rightarrow$ Boat wiggles left in phase 1 and wiggles right in phase 2

With all the forces of physics that go into pushing the boat and each rower's different ability, it can be easy for this to happen.

If the rowers use their body to balance the wiggle then they would be wasting energy that could have been used to make the shell go faster.

All this information comes flooding back when you realize that since Dad's not here nobody will be on the boat acting as a coxswain. All the moving recently must have made him forget the big day.
Luckily the other team's Coxswain also couldn't make it today, probably because their coxswain is your teacher who's still figuring out how to wash out the triangles from your desk.

Unfortunately, this means it's even more likely for your shell to wiggle now because you don't have someone who can correct your movements and make your team swing simultaneously.

It also doesn't help that the opposing crew is older and more experienced. It's a big problem because half your crew, including you, started last week.

You realize that the only way you can win is to find a way to minimize wiggle. You remember that just like in class when you were trying to prevent a tower from toppling, there might be a similar way to prevent the shell from tilting.

The only way to do that is by rearranging the riggings on the shell.

Luckily, you happened to have a handy dandy physics equation that your Dad gave you once, this equation lets you calculate the "moment" or the amount of wiggle of the shell based on the arrangement of the oars.

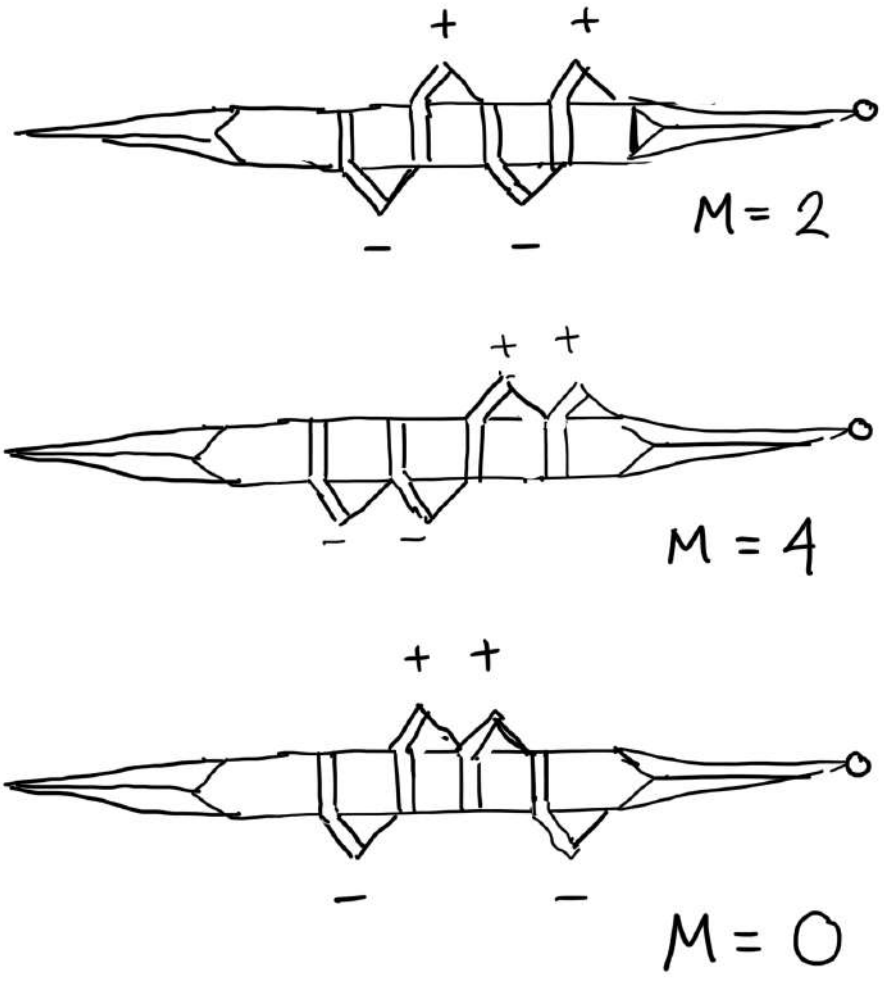
$$M = sF \pm (s+x)F \pm (s+2x)F \pm (s+3x)F$$

F, s, and x are physics variables that represent the individual variables of each rower. They'll just be a value of 1 since the difference in skill on your team is negligible. Rearranging the equation from here you get:

$$M = \pm 1 \pm 2 \pm 3 \pm 4$$

Where plus or minus is determined by the position of the rigging.

Given the need for an equal number of oars on each side, you figure that there are 3 distinct rigging patterns (excluding the riggings that are just flipped versions of these).



Both teams are currently rowing their 4x in an alternating pattern as shown by the first shell. If you plug that pattern into the Moments equation you find that it has a moment of 2

You assign the equivalent direction of the rigging to a corresponding positive or negative value in the moment equation.

The last possibility gives a moment of zero or zero wiggle. You and your team quickly rearrange the riggings on your boat to match that pattern while the other team laughs.

$$M = -1 + 2 + 3 - 4$$

$$M = 0$$

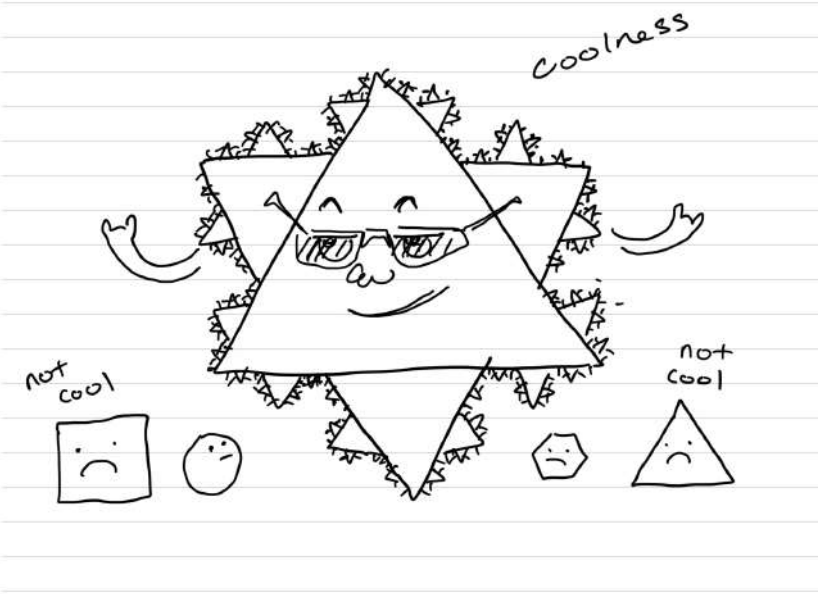
The race starts, and the energy lost to wiggle by the other team helps you close the gap. With no coxswain present, and with big egos, they thought they could make up for the wiggle of their boat by shifting.

However, even if this seemed like a good idea, unlike you, none of them seemed to know how to work as a team. When their shell began wiggling, everyone on their team shifted their bodies in different ways, making their boat wiggle even more and finally tip over. Your team left them in the dust.

Dad drives you back to Mom's to fetch some stuff so you can stay at his place for the weekend. He picks your sister up along the way. You get out your notebook to doodle and pass the time as the sea makes its way into view alongside the redwoods and squawking gulls. The sun is just beginning to set, painting the landscape in gold. The smell of ocean salt fills the breeze.

Because of your originality, you finish where you left off and draw more triangles.

To switch things up you draw a big triangle with equal sides and continue with three smaller triangles on its sides. On each edge that remains you continue to do the same thing, making sure the triangles are evenly spaced.

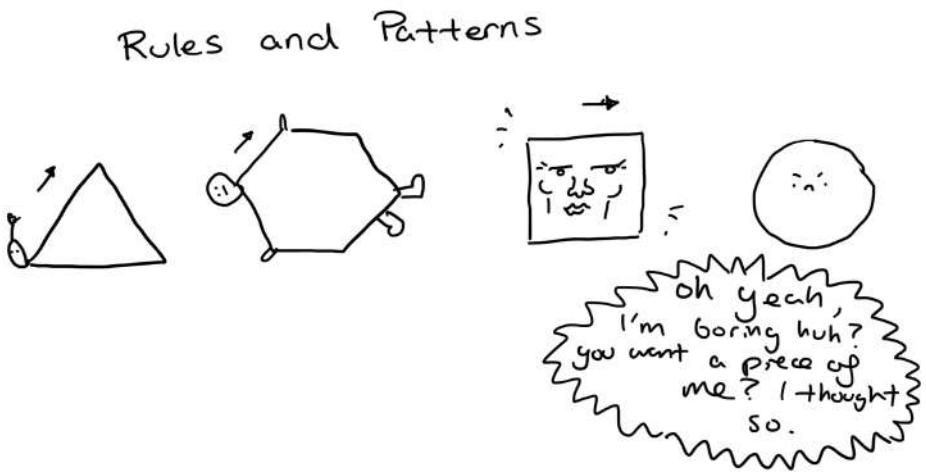


You get a pretty cool pattern, but your hand's getting tired of drawing triangles so you go back to drawing zig-zags again.

You realize that if you start to make rules for your ziggly zags and your zaggly zigs you get different patterns.

You come up with a few but all your rules just seem to get repetitive again and go back on themselves forming anything between a triangle or a circle.

If only there was some pattern that wasn't so underwhelming and boring.

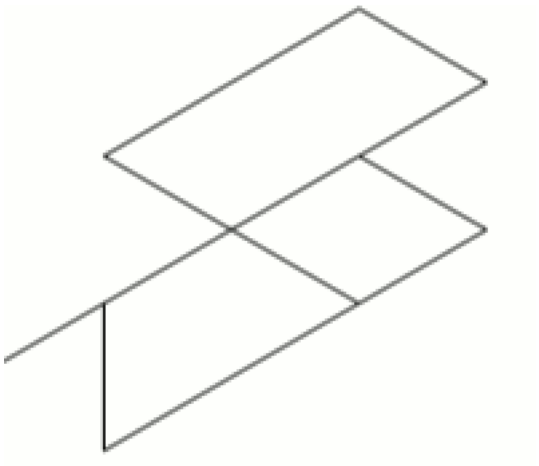


It seems obvious then to try the pattern that you learned today. You write down the first 32 items in the pattern.

ABBABAABBAABABBA BABABBABBAABBAAB

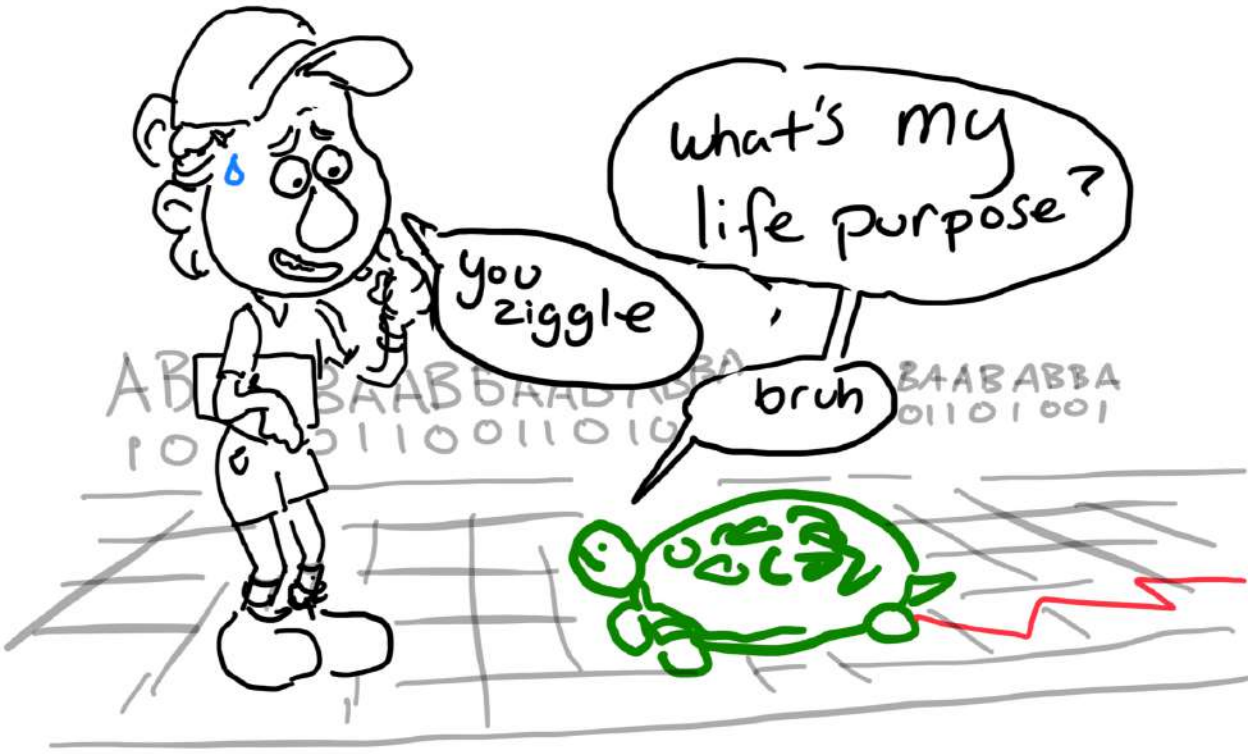
You try again, this time following the rule that every time you see an "A" you make a line and every time you see a "B" you don't move but turn the direction of your line by 60 degrees.

This is the shape you get after 32 moves. It's unique because it's the first that doesn't always resemble something between a triangle and a circle.



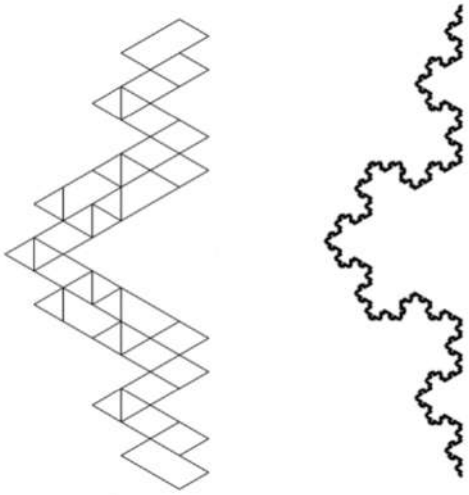
You wonder what the pattern will look like if you keep going but it looks like that's gonna take a while and you're running out of space on your notebook.

You tap your sister's shoulder in the front seat. She rolls her eyes and reluctantly hands you her laptop. You open your sister's turtle app which is a program that does the zig-zagging for you instead of your hand as long as you tell it what to do.



You tell the turtle to follow your ABBA pattern where you turn 60 degrees every time it encounters a B.

Since you've already made 32 moves and 32 is just 2^5 (2 multiplied by itself 5 times), you decide to experiment with higher powers.



Here's what you got with 2^8 and 2^{14} moves respectively.

You change your rule up a bit and get these other patterns.

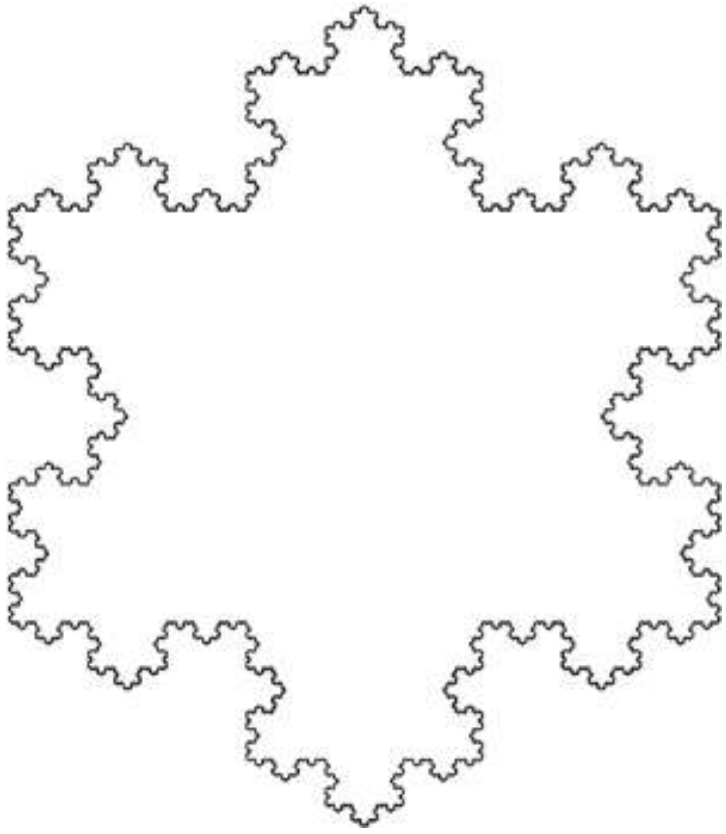
Honey Comb:
A
B

Sun and Stars:
A
B

A quick Google search and you realize what you've been following the entire day has been the "Prouhet–Thue–Morse Sequence" or PTM Sequence, which is put into your turtle for an infinite amount of steps to approach the "Koch Curve".

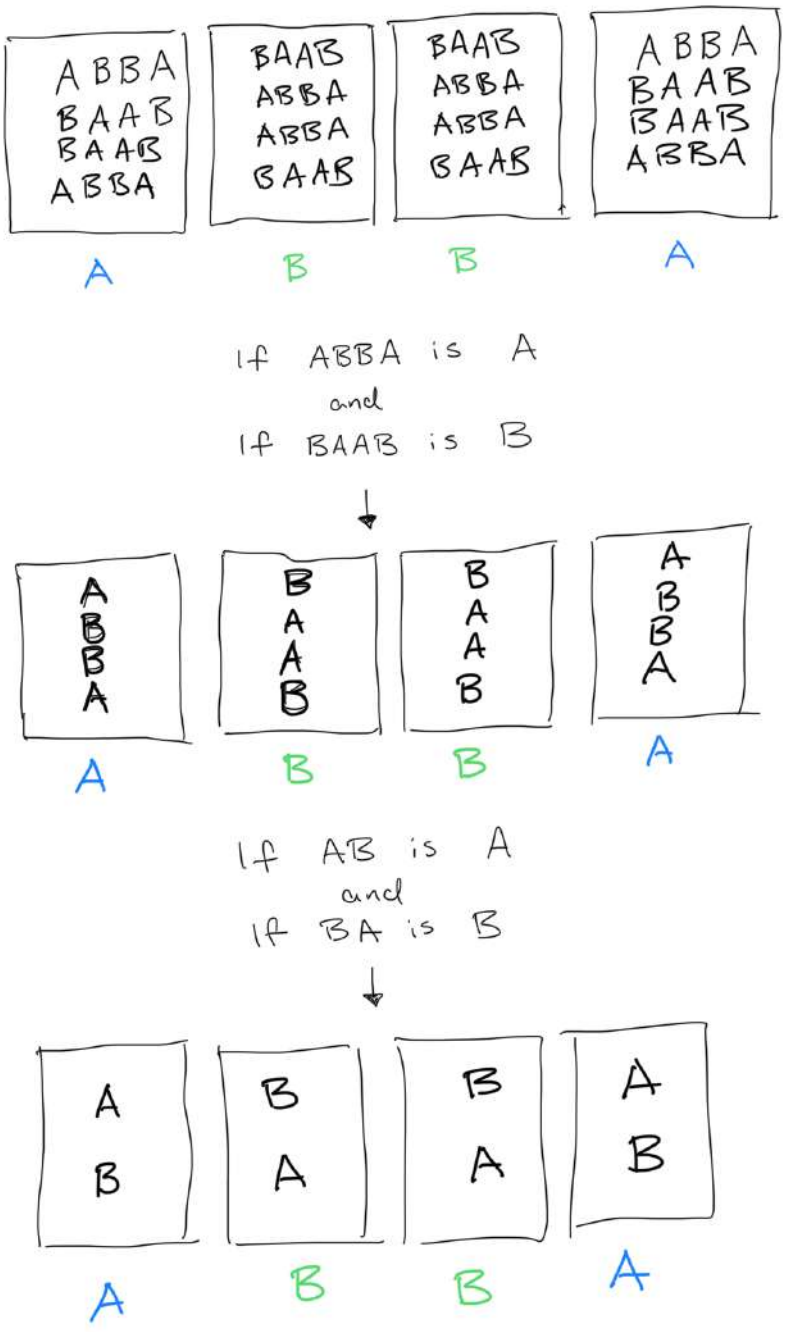
What you're getting isn't just any ol' pattern but a fractal, a shape that contains itself an infinite amount.

By putting these Koch curves together you would get this fractal shape.



If you zoom in on any part of the snowflake you'd just see the Koch Curve Repeated because this pattern contains itself an infinite number of times.

To find out why you look back at your PTM sequence in letter form. By grouping the letters in 4's, and 8's, you find that the PTM Sequence repeats itself over and over.



The car arrives at Mom's house, and you race your sister to the door while Dad waits outside. Inside, you find your mom who looks quite upset. She tells you that she doesn't like sharing custody of you and your sister with Dad, especially with the weekly changes.

As a professional astrologer, repeating a pattern three times in a row lays down a permanent routine, and she can't stand her days spent with you being enforced by such a rigid alternating routine.

You suggest that the TPM Sequence could be a great way for them to share custody without a fixed pattern, so neither of them would feel like the other has more custody, and nothing would repeat three times. The same goes with their property.

You hug Mom before running out the door with your stuff.

Dad parks in a nearby McDonald's since the motel parking is all full tonight. Your sister's crying because her boyfriend just texted her to end things. He's dating her friend now. It's alright because he'll break up with that girl next week, then date another one of your sister's friends, before breaking up with her and getting back together with your sister again.

Dad decides to get ice cream to cheer her up. As you guys walk to McDonald's you recall the day's events and how excited you are to see Dad's new place.

You listen to the sobs of your sister as they're accompanied by a symphony of buzzing late-night neon lights and ringing registers. You tell Dad that you think you want to quit rowing.

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