# *Understanding Fuel Consumption: The Effect of the Weather on Aviation*

(Jonathan Solomon)

## *1. Introduction*

We all experience the weather, whether on our daily commutes or while playing sports. It engulfs us, shaping our routines and influencing key decisions. No one wants to climb a mountain during a hurricane or play rugby when the pitch is frozen solid. Beyond this, accurate weather forecasting is essential for industries such as agriculture, where farmers depend on rainfall predictions for crop planning, and aviation, where optimal flight paths rely on wind patterns and storm avoidance. It is the aviation part of this I want to focus on, creating a model for the fuel consumption of a plane based on factors such as wind and turbulence.

Despite remarkable advancements in meteorology, predicting the weather remains an inherently complicated issue, unless you are in Britain where it is always raining. Our atmosphere, key to our survival is extremely difficult to understand. Fundamentally we can describe the atmosphere through complex sets of equations describing fluid motion, the transfer of energy and thermodynamics. Whilst these equations are widely used and understood, they are extremely sensitive based on our initial conditions. This was identified by Edward Lorenz who in the 1960s wrote about the butterfly effect, the idea that tiny variations can lead to vastly different outcomes. That itself is a whole other assortment of problems which I would love to get into but for this essay we will be focussing on modelling fuel consumption.

## *2. The Importance of Accurate Prediction in Aviation*

When it comes to flying, the weather is dominant. Winds control the routes across the sky in which we fly our planes, turbulence is not only a factor for comfort yet has a surprisingly substantial impact on fuel consumption.

When it comes to fuel consumption minimising the costs for airlines is just as important for increasing their profit as it is for keeping ticket prices low. As fuel costs rise across the world, so do operating costs; hence as do our ticket prices.

### *2.1 Fuel Consumption: A Windy Relationship*

This isn't going to be a breeze to explain, so prepare for some fasten your seatbelt and prepare for some turbulence.

Let:

$$v = cruise\ speed\ (kmh^{-1})$$

$$F = fuel\ consumption\ in\ still\ air\ (Lkm^{-1})$$

$$w = wind\ speed\ (kmh^{-1})$$

$$v_{eff} = actual\ ground\ speed\ (kmh^{-1})$$

If we assume $w$ to be either a direct headwind or tailwind, we can simply define this equation.

$$v_{eff} = v - w \ (1)$$

For a constant indicated airspeed[1], fuel burn is essentially constant. Therefore, fuel burn per hour remains constant however the aircraft will travel a different distance in that time. Hence, we can define the effective fuel burn per kilometre as:

$$F_{eff}(w) = \frac{Fv}{v - w} = \frac{Fv}{v_{eff}} \ (2)$$

Now we could leave this formula as it is, but what is the fun in that. Why don't we take the Taylor expansion of this function. We can rewrite equation (2) as:

$$Fv \cdot (v - w)^{-1} \ (3)$$

Now the Taylor series for $(1 - x)^{-1}$ is around 0:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n \ (4)$$

$$f(x) = (1 - x)^{-1} \therefore f(0) = 1 \ (5)$$

$$f'(x) = (1 - x)^{-2} \therefore f'(0) = 1 \ (6)$$

$$f''(x) = 2(1 - x)^{-3} \therefore f''(0) = 2 \ (7)$$

$$f'''(x) = 6(1 - x)^{-4} \therefore f'''(0) = 6 \ (8)$$

Hence substituting these values into equation (4) gives the Taylor series (Maclaurin series) of $(1 - x)^{-1}$ as:

$$\sum_{n=0}^{\infty} x^n \ (9)$$

Now letting $x = \frac{w}{v}$ we get the equation:

$$\frac{1}{1 - \frac{w}{v}} = 1 + \frac{w}{v} + \frac{w^2}{v^2} + \cdots + \frac{w^n}{v^n} \ (10)$$

Multiplying by $\frac{1}{v}$:

$$\frac{1}{v - w} = \frac{1}{v}\left(1 + \frac{w}{v} + \frac{w^2}{v^2} + \cdots + \frac{w^n}{v^n}\right) \ (11)$$

Now the original equation (2) has a multiple of $Fv$ so, we can multiply (11) by $Fv$ to give:

$$F_{eff}(w) = F\left(1 + \frac{w}{v} + \frac{w^2}{v^2} + O(w^3)\right) \ (12)$$

[1] The raw airspeed of an aircraft indicated on the pilots' primary flight display, it is calculated from the pitot-static port system

However, the wind is never this convenient and given our assumption that $w$ is either a direct headwind or tailwind this formula alone is not particularly useful. We can fix this with some quite simple trigonometry.

Let:

$$\theta = angle\ of\ wind\ from\ nose\ of\ plane$$

$$m = magnitude\ of\ wind\ speed\ (kmh^{-1})$$

$$w = wind\ speed\ in\ relation\ to\ direction\ of\ plane\ (kmh^{-1})$$

Since we want $w$ w we can create a simple right-angled triangle where $m$ is the hypotenuse and $w$ is the adjacent side to the angle. Therefore $w$ is given by:

$$w = m \cdot \cos(\theta),\ 0 \leq \theta \leq 2\pi\ (13)$$

## *2.2 Application of Our Formula*

In the previous section we defined equation (12) now I want to apply this to some real-world numbers. Take a Lufthansa B787-9 which consumes a quoted value[2] of 2.5 litres of kerosene per passenger per 100km flown. Given that this plane has 294 seats:

$$2.5 \cdot 294 = 735L\ per\ 100km$$

$$\frac{735}{100} = 7.35L\ per\ km$$

Now if we take the cruise speed to be 913 km per hour with our plane flying on a bearing of 270° with winds of 100 km per hour from a bearing of 280° hence the angle between the plane's nose and the direction of wind measured clockwise is 10°. Therefore, we can deduce that this is true:

$$w = 100 \cdot \cos(10°) \Rightarrow w \approx 98.48\ kmh^{-1}$$

Since $F = 7.35\ Lkm^{-1}$ and $v = 913\ kmh^{-1}$:

$$F_{eff}(98.48) \approx 7.35\left(1 + \frac{98.48}{913} + \frac{98.48^2}{913^2} + \frac{98.48^3}{913^3}\right)$$

$$F_{eff}(98.48) \approx 8.24\ Lkm^{-1}$$

This represents a 12.1% increase in fuel consumption with a 100kmh-1 headwind hence an exceptionally significant increase in cost for airlines. It is this increase in cost which highlights the necessity for accurate prediction for planning routes to minimise the increase of fuel needed to travel the same distance.

To better present this I have created a 3D modelling of this formula in python below, in Figure 1 we see the effect when the planes speed is 750 $kmh^{-1}$ and in Figure 2 we do see that lower speeds seem to have an increased proportion of fuel required given that the base consumption to maintain that cruise speed is the same. This model may have slight inaccuracies

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2 https://www.lufthansagroup.com/en/themes/boeing-787-9.html#:~:text=Ultra%2Dmodern%20long%2Dhaul%20aircraft,passenger%20per%20100%20kilometers%20flown.

since a steady fuel consumption of 7.35 L per km may not be required to maintain the lower speeds however, I doubt this has a considerable effect on my point.

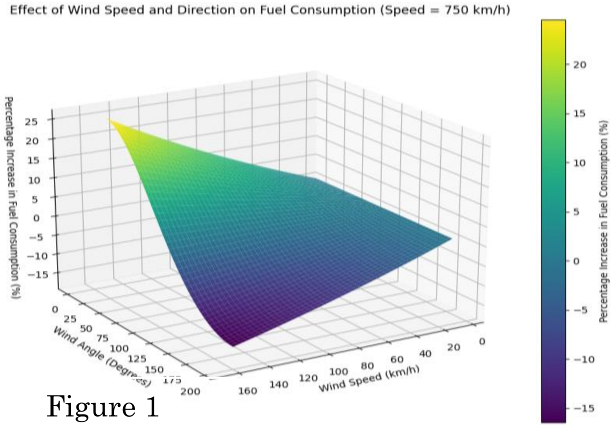


Figure 1

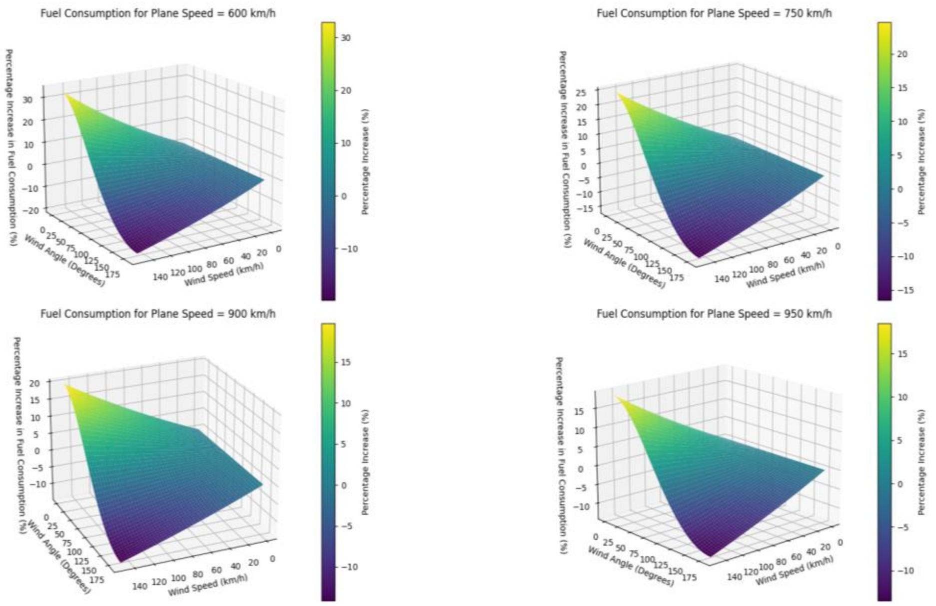


Figure 2

## *2.3 Modelling Turbulence*

Wind is never constant, it is always shifting and changing with time. In response to air pressure, storms or just generally when we wake up on the wrong side of the bed. So, to incorporate turbulence we must model it as a random process or stochastic.

Let:

$$w(t) = actual\ windspeed\ at\ time\ t$$

$$\mu_w = mean\ wind\ speed$$

$$\sigma_w = deviation\ in\ wind$$

$$\xi(t) \sim N(0,1) = random\ variable\ giving\ random\ gusts$$

Such that:

$$w(t) = \mu_w + \sigma_w \xi(t) \ (14)$$

It is this randomness which introduces a lot of complexity into our modelling, we can no longer assume the windspeed to be a constant and have now defined it as a function which changes with time. We can now further include this into our other equation (12) and equation (2).

Since w is now stochastic so is fuel consumption hence:

$$F_{eff}(\mu_w, \sigma_w) = \frac{F \cdot v}{v - (\mu_w + \sigma_w \xi)} \ (15)$$

Given that the expected value of the normal distribution is 0 we can find the expected value of the fuel efficiency.

$$E\left[F_{eff}(\mu_w)\right] \approx \frac{F \cdot v}{v - \mu_w} \ (16)$$

Note that this is the same as equation (2) just in terms of mean wind speed rather than actual windspeed. However major differences arise when we take the Taylor expansion around $v = \mu_w$ so let us do that:

$$F_{eff}(w) \approx F_{eff}(\mu_w) + F'_{eff}(\mu_w)(w - \mu_w) + \frac{F''_{eff}(\mu_w)}{2}(w - \mu_w)^2 \ (17)$$

$$F'_{eff}(w) = \frac{Fv}{(v - w)^2} \ (18)$$

$$F''_{eff}(w) = \frac{2Fv}{(v - w)^3} \ (19)$$

Substituting equations (18) and (19) into (17) and given that $E[w - \mu_w]$ is 0 since the expected deviation from the mean is 0 and $E[(w - \mu_w)^2] = \sigma_w^2$ because variance is defined as the expectation of the deviation from the mean squared. We now have this:

$$E\left[F_{eff}(w)\right] \approx \frac{Fv}{v - \mu_w} + \frac{Fv\sigma_w^2}{(v - \mu_w)^3} \ (20)$$

Now what does this mean? The second term in equation (20) with a cubic denominator shows how the effect the wind has on the fuel consumption is not linear as the mean wind speed approaches the actual speed of the aircraft, the denominator rapidly shrinks causing a sharp spike in fuel consumption. This is reflective of the real world, with a higher wind speed, more fuel is needed to be burnt to reach the same destination. And as that wind speed becomes much higher, a much greater amount of fuel is required. This second term also considers variability in the wind speeds with the use of variance hence capturing the uncertainty we have with winds.

## *2.4 Increasing the Models Accuracy*

Now given that we have created equation (20) there could be a huge issue as $\mu_w \rightarrow v$ because fuel consumption could theoretically become infinite due to the large impact of the denominator in our model so we can introduce some logarithmic corrections.

$$E[F_{eff}(w)] \approx \frac{Fv}{v-\mu_w} + \frac{Fv\sigma_w^2}{(v-\mu_w)^3} + F\log\left(1-\frac{\mu_w}{v}\right) \quad (21)$$

We can also add a second order correction, which arises from the Taylor expansion to this giving:

$$E[F_{eff}(w)] \approx \frac{Fv}{v-\mu_w} + \frac{Fv\sigma_w^2}{(v-\mu_w)^3} + F\log\left(1-\frac{\mu_w}{v}\right) + \frac{F\sigma_w^2}{2v^2\left(1-\frac{\mu_w}{v}\right)^2} \quad (22)$$

### *2.5 Applying Numbers*

Now before moving onto finding a final expression for the total fuel burn of a flight, I wanted to stop for a second and apply some numbers to this expression. Note these are just example numbers and can be changed depending on the specific conditions.

$$F = 7.35\, Lhm^{-1}$$

$$v = 900\, kmh^{-1}$$

$$\mu_w = 120\, kmh^{-1}$$

$$\sigma_w^2 = 25\, kmh^{-1}$$

$$\theta = 25°$$

Now calculating the effective values for the mean wind speed and variance gives:

$$\mu_{w(eff)} = 120 \cdot \cos(25°) \approx 108.76$$

$$\sigma_{w(eff)}^2 = 25 \cdot \cos^2(25°) \approx 20.53$$

Using this and the numbers I gave earlier we can find a value for the expected fuel consumption.

$$E[F_{eff}(w)] = 8.36 + 2.74 \cdot 10^{-4} - 0.411 + 1.21 \cdot 10^{-4}$$

$$E[F_{eff}(w)] = 7.95\, Lkm^{-1}\,(2dp)$$

This represents a percentage increase of 8% for these values showing the exceptionally substantial impact that wind has on a flight, and this seems like a very reasonable answer for medium wind speeds such as these.

### *2.6 Finding Total Fuel Burn*

To find the total fuel burn of a flight we need the distance of the flight and equation (22) for the expectation of the fuel efficiency at any given moment in time. From this we can integrate with respect to time. Let:

$$F_{total} = total\, fuel\, burn\, (L)$$

$$D = length\, of\, flight\, (km)$$

$$T = time\, of\, flight\, (hrs)$$

This gives:

$$F_{total} = \int_0^T E[F_{eff}(w)]v_{eff}\,dt \quad (23)$$

$$v_{eff} = v - \mu_w \cos(\theta) \quad (24)$$

$$T = \frac{D}{v_{eff}} = \frac{D}{v - \mu_w \cos(\theta)} \quad (25)$$

Now this could be quite difficult to evaluate by itself so by letting:

$$dx = v_{eff} dt \quad (26)$$

$$\therefore\ dt = \frac{dx}{v_{eff}} \quad (27)$$

Hence:

$$F_{total} = \int_0^D E[F_{eff}(w)]dx \quad (28)$$

$$F_{total} = \left[E[F_{eff}(w)]x\right]_0^D \quad (29)$$

And therefore finally:

$$F_{total} = D \cdot \left[\frac{Fv}{v - \mu_w} + \frac{Fv\sigma_w^2}{(v - \mu_w)^3} + F\log\left(1 - \frac{\mu_w}{v}\right) + \frac{F\sigma_w^2}{2v^2\left(1 - \frac{\mu_w}{v}\right)^2}\right] \quad (30)$$

$$F_{total} = \frac{DFv}{v - \mu_w} + \frac{DFv\sigma_w^2}{(v - \mu_w)^3} + DF\log\left(1 - \frac{\mu_w}{v}\right) + \frac{DF\sigma_w^2}{2v^2\left(1 - \frac{\mu_w}{v}\right)^2} \quad (31)$$

Now we can finally apply numbers to this final formula for the total fuel burn of a flight which accounts for multiple different scenarios including turbulence and other corrections.

So, say:

$$D = 5500\ km$$

$$F = 7.35\ Lkm^{-1}$$

$$v = 913\ kmh^{-1}$$

$$\mu_w = 80\ kmh^{-1}$$

$$\sigma_w^2 = 20\ km^2h^{-2}$$

$$\theta = 20°$$

As with the calculations in section 2.5 we need to calculate the effect of the wind relative to the direction of the wind, hence:

$$\mu_{w(eff)} = 80 \cdot \cos(20°) \approx 75.18$$

$$\sigma_{w(eff)}^2 = 20 \cdot \cos^2(20°) \approx 17.66$$

Combining this are our other numbers and putting into equation (31), we can find a value for the total fuel consumption of this flight.

$$F_{total} = \frac{36908025}{(913 - 75.18)} + \frac{5500 \cdot 7.35 \cdot 913 \cdot 17.66}{(913 - 75.18)^3} + 40425 \cdot \log\left(1 - \frac{75.18}{913}\right) + \frac{5500 \cdot 7.35 \cdot 17.66}{2 \cdot (913)^2 \left(1 - \frac{75.18}{913}\right)^2} F_{total}$$

$$= 44052.4516 + 1.108305109 - 1508.663188 + 0.5085214602$$

$$F_{total} = 42545.41L\ (2dp)$$

This represents an increase of 2080.41L over the 5500km journey due to the impact of this wind, and the variability described by the turbulence which we account for using the variance. What I find also important to note is that the logarithmic term will produce a negative value this is because to use the logarithm we must normalise by dividing by v. This means that each of the terms have no units. Now in a practical setting all this term does is correct for overestimation from the other terms.

## *3. Conclusion*

In this essay I have created a model which demonstrates the effect of weather on aviation. Specifically, the impact of wind and turbulence. Developing this model which accounts for the velocity of the plane, wind velocity and angles has not only just been a mathematical exercise for this essay. It has also had a profound impact on me for understand the necessity of minimising costs in aviation, as an onlooker to the industry. Fuel itself has extortionate costs that we see on a day-to-day basis however when we are talking about the scale and volumes of fuel planes require, slight changes in wind direction have an enormous impact.

We also expanded the model to account for the stochastic impact of wind and turbulence, considering the randomness which allows us to approximate values for fuel consumption using the Taylor expansion and probability.

Overall, this essay highlights the critical importance of predicting the weather. For airlines to be able to plot courses which minimise the mean wind speeds and variability which leads to not only more efficient flight yet a reduction in fuel costs and a lower environmental impact.