Where is the Virtual Image of an Underwater Object

Yushi Huang

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1 Introduction

Before entering high school and studying physics formally, a lot of us already know the very basic fact that objects underwater seem to be shallower than their actual depths. When we want to see objects at the bottom of an opaque cup but find the depth of the cup tall enough to obstruct light beams from the bottom, we can probably see the objects if we fill the cup with water, a medium with a refractive index larger than 1. Also, when catching fish in the river using a spear, common experiences tell us to aim below the fish to hit its actual location. After studying the definition of refraction in the high school physics course, we have a much more solid understanding on why the location of the virtual image we perceive for an underwater object is always higher than the real location. However, a more nuanced question to this conventional physics model arises after I have watched an inspiring video on *bilibili*: is the virtual image of an underwater object further or closer to the observer along the horizontal direction?

As I scanned through some popular physics textbooks, there seems to be no consensus to the answer of this problem. As shown in the following examples, three completely different visualizations occur: the virtual image does not move, stands closer, and stands further compared to the observer, making this problem more captivating and valuable to explore. Since we know that the answer to this problem is sure to be definite in this natural world, we seek to find it out in this essay and discuss important theorems and relations in geometric optics using GeoGebra, a powerful computational tool for geometric visualization.

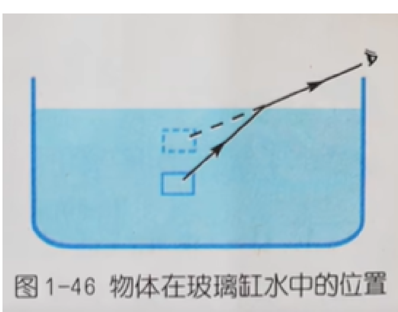

Figure 1: Remain

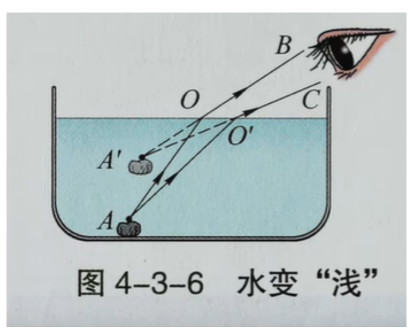

Figure 2: Closer

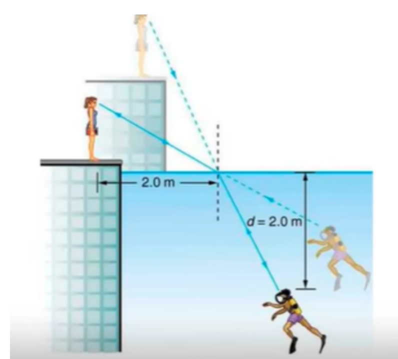

Figure 3: Further

2 Demonstration of Snell's Law

Snell's law is a pivotal geometric relation between the angles of incidence and refraction, when referring to light or other waves passing through a boundary between two media (in this essay we always refer to air and water). It states that

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

where α is the angle of incidence, β is the angle of refraction, v_1 is the speed of the beam in the air, and v_2 is the speed of the beam in the water. We first present a precise mathematical proof of this relation using Fermat's principle in optics.

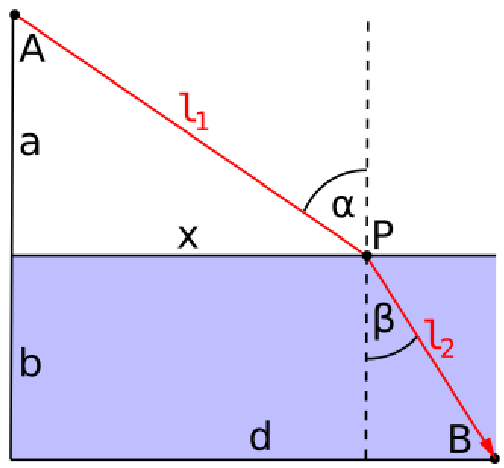

Figure 4: Refraction

Fermat's principle, also known as the principle of least time, is the link between ray optics and wave optics. Fermat's principle states that the path taken by a ray between two given points is the path that can be traveled in the least time. We first represent the total time the light travels using the known variables.

$$t = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{(d-x)^2 + b^2}}{v_2}$$

Using basic knowledge of calculus, in order to find the conditions for minimized t, we calculate the first derivative of t over x and make it zero.

$$\frac{dt}{dx} = \frac{x}{v_1\sqrt{a^2 + x^2}} + \frac{x-d}{v_2\sqrt{(d-x)^2 + b^2}} = 0$$

Since

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{x}{l_1} = \sin\alpha$$

and

$$\frac{x-d}{\sqrt{(d-x)^2 + b^2}} = \frac{x-d}{l_2} = -\sin\beta$$

we get Snell's law

$$\frac{\sin\alpha}{v_1} - \frac{\sin\beta}{v_2} = 0 \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}$$

The refractive index of a medium is defined as,

$$n = \frac{c}{v}$$

where c is the speed of light in vacuum, and v is the speed of light in a medium. Thus, Snell's law can be written as,

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1}$$

3 Refraction

With the tool of Snell's law, we derive the location of the virtual image of an underwater object in this section. Powered by GeoGebra, you can access this file to explore the location on your own https://www.geogebra.org/classic/phajwvpa.

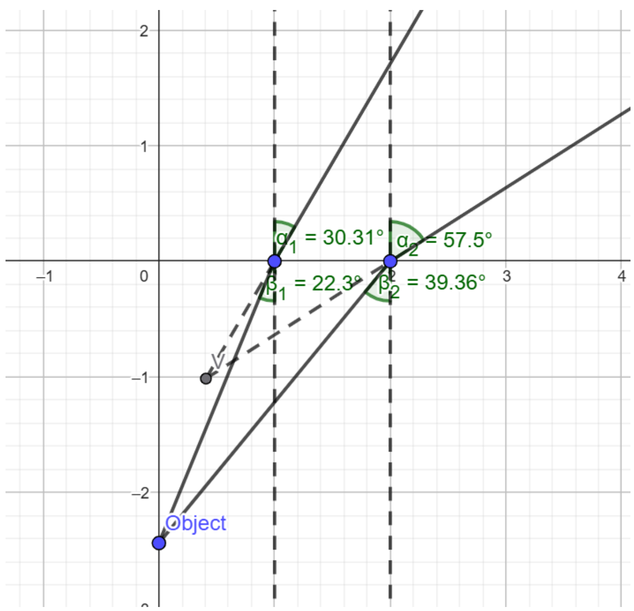

Figure 5: Two Refracted Rays

In this image, assume the object is c units underwater, making its coordinate $(0, -c)$. The two light beams from the object intersect with the water surface at point A $(a, 0)$ and B $(b, 0)$, respectively $(a < b)$. Next, we derive the coordinate of its virtual image V using Snell's law. The sine value of the angle of incidence at A can be calculated as,

$$\sin\beta_1 = \frac{a}{\sqrt{a^2 + c^2}}$$

Snell's law gives the angle of refraction,

$$\sin\alpha_1 = n\sin\beta_1 = \frac{na}{\sqrt{a^2 + c^2}}$$

$$\tan\alpha_1 = \frac{na}{\sqrt{a^2 + c^2 - (na)^2}}$$

Similarly, through point B,

$$\tan\alpha_2 = \frac{nb}{\sqrt{b^2 + c^2 - (nb)^2}}$$

Suppose the perpendicular distance between V and the normal line through A is x. We establish the equation,

$$\left(\frac{x}{\tan\alpha_1}\right)\tan\alpha_2 = x + (b - a)$$

$$\begin{aligned}
x &= \frac{b-a}{\frac{\tan\alpha_2}{\tan\alpha_1} - 1} \\
&= \frac{b-a}{\frac{b\sqrt{c^2-(n^2-1)a^2}}{a\sqrt{c^2-(n^2-1)b^2}} - 1} \\
&< \frac{b-a}{\frac{b\sqrt{c^2-(n^2-1)b^2}}{a\sqrt{c^2-(n^2-1)b^2}} - 1} \\
&= \frac{b-a}{\frac{b}{a} - 1} \\
&= a
\end{aligned}$$

Since the perpendicular distance between the virtual image and the normal line is strictly shorter than that between the underwater object and the normal line, the virtual image of the object is always closer to the observer than its real location, except for the situation when $a = b$ (the observer is directly on top of the object).

This mathematical calculation proves that the virtual image of an object under the water is always closer to the observer compared with its actual location. However, when putting a straight ruler vertically in a container filled with water, the image under the water surface does not seem to be closer to human eyes, as shown in Figure 6 (this image comes from the video mentioned before). Is this a contradiction with the conclusion we have derived before using mathematical and physical calculations? Is it possible that the virtual image is just above the underwater object? Should we believe the theoretical derivations or our human observation?

Figure 6: Ruler in the water

4 Experiment

These unresolved mysteries motivate us to explore the truth using experiments. In the video, we use a telephoto lens, which is set to a very shallow depth of field so that it can precisely focus on things at the focal point. When the image of an object captured by the telephoto lens is clear, we claim the focal length at this moment as the distance between the lens and the object. The accuracy of the telephoto lens makes it possible to locate the virtual image of underwater objects.

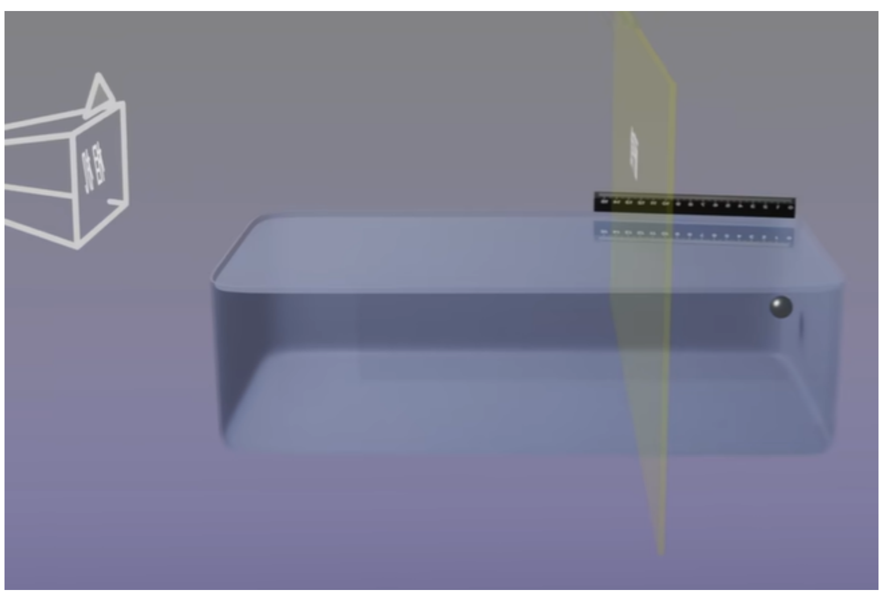

Figure 7: Experiment Setup

In Figure 7, the white box refers to the telephoto lens, and the yellow plane represents the distance of the focal length, also the location where we can capture clear photographs from the lens. A steel ball is placed at the back of the water container, along with a straight ruler. During the experiment, we extend the focal length of the telephoto lens. When the reflection point of the ball through the lens changes from blurred to clear, it means that we have focused on the virtual image of the ball in the water. Then, we check which calibration on the ruler is the clearest. This measurement determines the exact location of the ball's virtual image.

When moving the focus point inwards, the ball gradually changes from blurred to bright. At position 6 cm (calibration on the ruler), we find the virtual image of the ball.

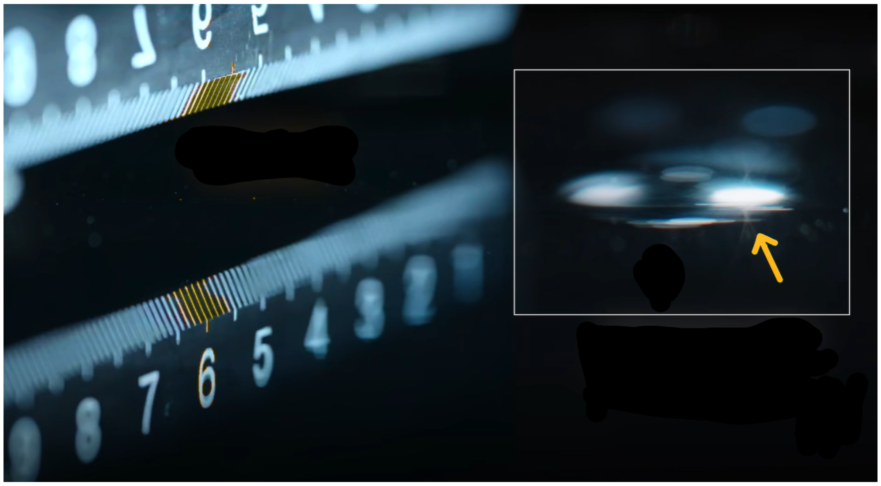

Figure 8: Virtual Image 1

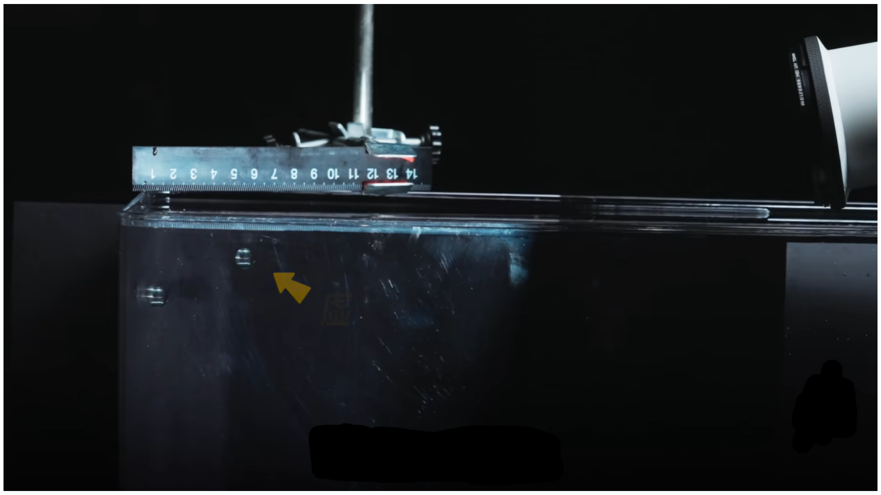

Figure 9: Location of Virtual Image 1

The position 6 cm is exactly in front of the actual steel ball, which perfectly aligns with the theoretical analysis we have done before. However, this is not the end of this experiment. When we continue to extend the focal length of the telephoto lens, an unexpected result occurs: we find another clear and bright image of the steel ball through the lens near position 1 cm, which is exactly above the actual steel ball.

Figure 10: Virtual Image 2

In addition, when we pay close attention to the orientations of the light spots in this two situations, we find out that the orientation in Figure 8 is horizontal, and the orientation in Figure 10 is vertical. Why are there so many intriguing phenomena from this experiment? Why we cannot find two virtual images from our mathematical model in Figure 5? What is the difference between the formation of these two virtual images? Is there any factor we've accidentally overlooked in this thought process?

5 Refraction in 3-D Space

It seems that we have fully resolved the refraction problem on the location of the virtual image relative to the real object in Figure 5. However, all of the above derivations ignore one simple yet important fact: instead of portraying light beams on a 2-dimensional plane, human beings perceive things underwater in a 3-dimensional space. This indicates that the light from the object should be a cone of light beams rather than 2 light beams on the same plane.

As a result, by doing online research and establishing a valid 3-dimensional mathematical model using GeoGebra, we present a more comprehensive demonstration to the two virtual images of underwater objects. Specifically, Figure 8 shows the meridional image, and Figure 10 shows the sagittal image.

5.1 Meridional Image

In Figure 11, the meridional image is formed by the extension lines of two light beams through A and B on the same plane. By moving A and B horizontally along the direction of x-axis, the meridional image presents a V shaped curve in the 3-dimensional space, shown in Figure 12.

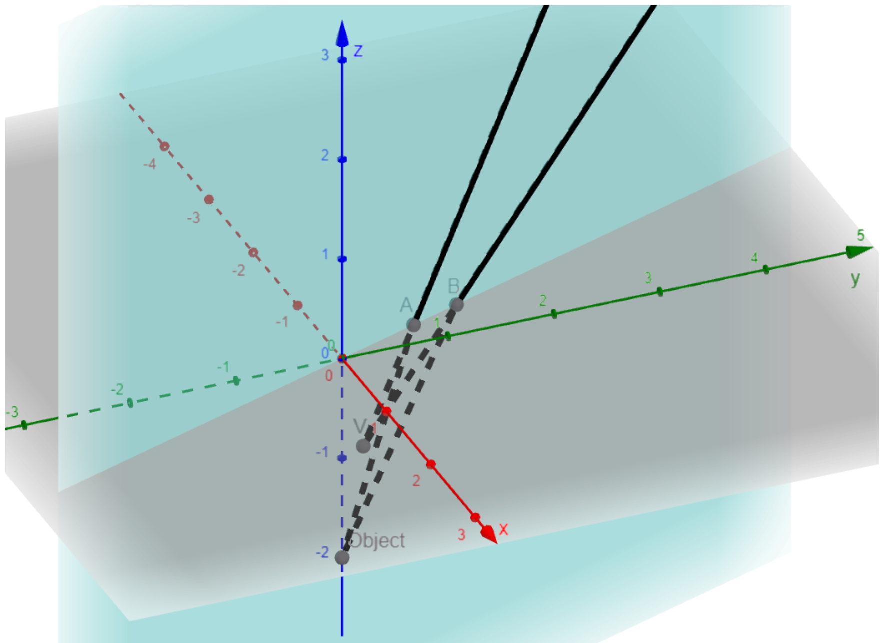

Figure 11: Meridional Image

When observing the meridional image along the direction of light beams, the image appears to be a roughly straight line, as shown in Figure 13. This aligns with our observation of the horizontal light spots in Figure 8.

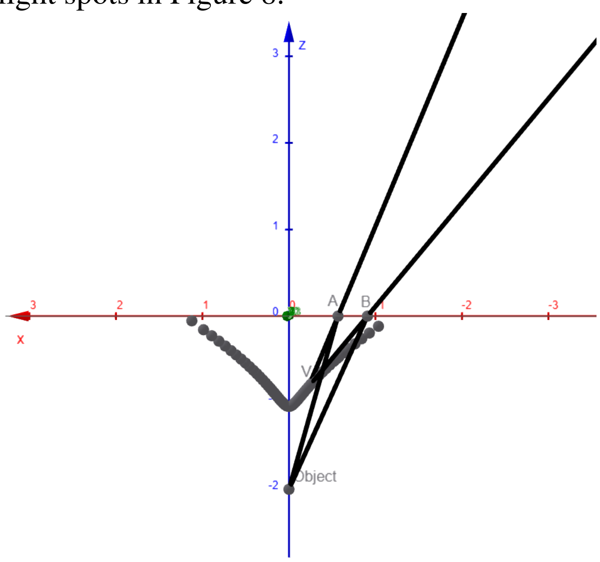

Figure 12: V Shaped Trace

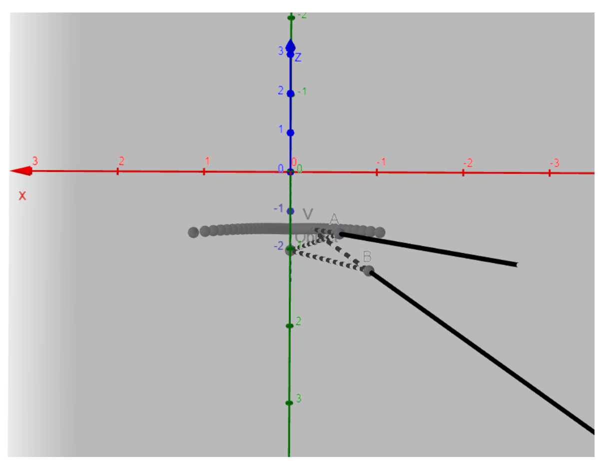

Figure 13: Line Shaped Trace

You can access this file to explore the location on your own https://www.geogebra.org/calculator/cyctvfhn.

5.2 Sagittal Image

In Figure 14, the sagittal image is formed by the extension lines of two light beams symmetric with respect to the y-axis.

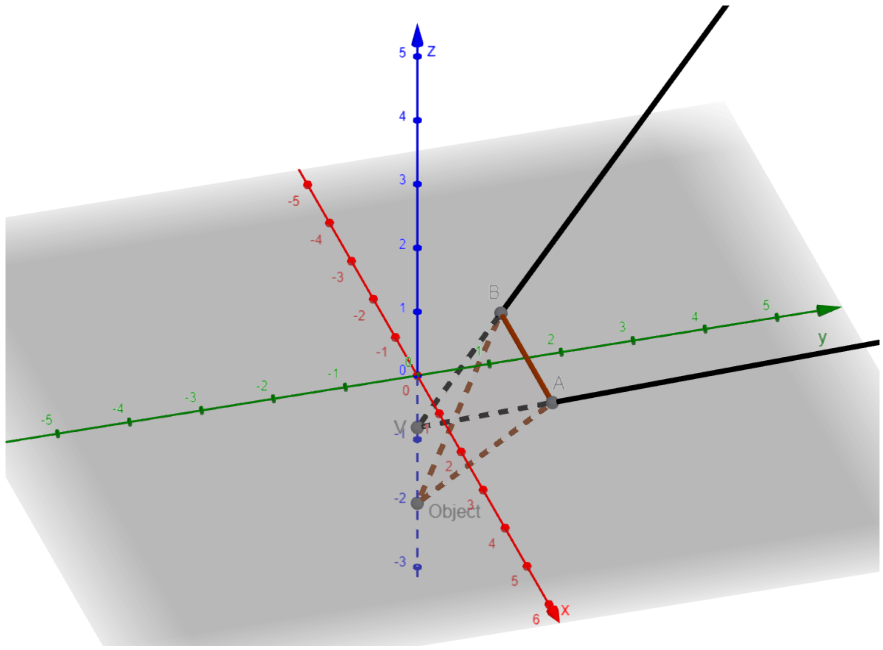

Figure 14: Sagittal Image

By changing the positions of A and B in GeoGebra, we record the trace of the virtual image point V. The shape of the sagittal image appears to be a vertical straight line, as shown in Figure 15. This aligns with our observation of the vertical light spots in Figure 10.

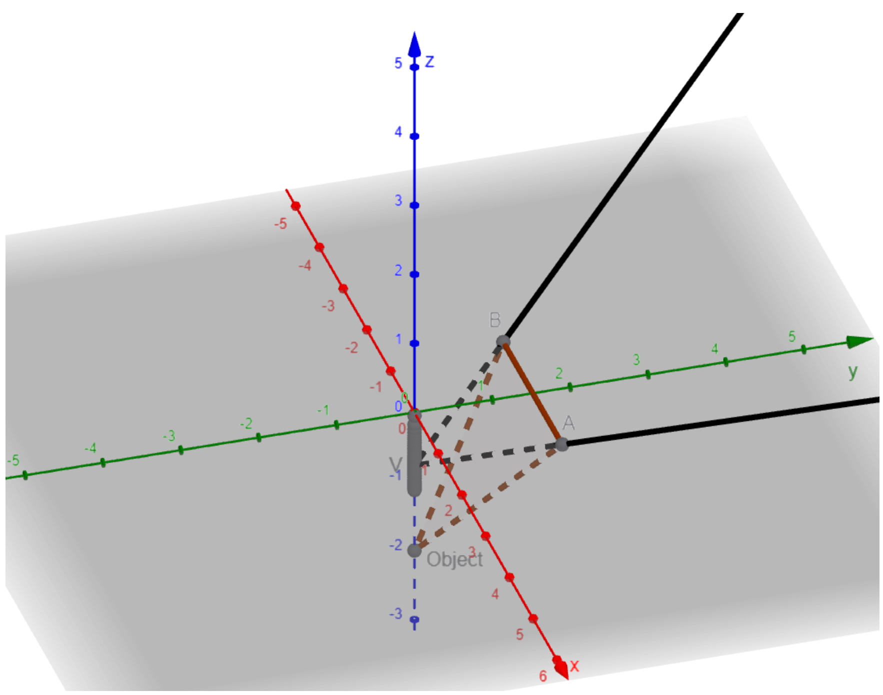

Figure 15: Vertical Line Trace

Next, we calculate the coordinates of point V in this model. Suppose the angle between the plane object-A-B and the z-axis is θ (angle of incidence), the angle between the plane V-A-B and the z-axis is α (angle of refraction), and the object is d units under the water surface. From the trace, we get to know that the sagittal image only changes its z coordinate. Using Snell's law,

$$\begin{aligned} z &= -\frac{\tan\theta}{\tan\alpha}d \\ &= -\frac{d\tan\theta}{n\sin\theta}\sqrt{1-(n\sin\theta)^2} \\ &= -\frac{d}{n}\sqrt{1+\tan^2\theta - n^2\tan^2\theta} \\ &= -\frac{d}{n}\sqrt{1-(n^2-1)\tan^2\theta} \end{aligned}$$

where n is the refractive index of water. When changing the positions of A and B, θ changes from 0 to $\arcsin\frac{1}{n}$ because otherwise a total internal reflection will occur.

$$\begin{aligned} \frac{dz}{d\theta} &= \frac{d}{2n\sqrt{1-(n^2-1)\tan^2\theta}} \cdot \frac{d}{d\theta}\left(1-(n^2-1)\tan^2\theta\right) \\ &= \frac{(n^2-1)d\tan\theta}{n\cos^2\theta\sqrt{1-(n^2-1)\tan^2\theta}} \\ &> 0 \end{aligned}$$

This indicates that z increases consistently when θ increases. Thus, the length of the sagittal image equals to maximum value of $|z|$, which is when $\theta = 0$, $|z| = \frac{d}{n}$.

You can access this file to explore the location on your own https://www.geogebra.org/classic/bdxudng6.

6 Conclusion

Since the meridional image stretches the image of the object horizontally, and the sagittal image stretches the image of the object vertically, the superposition of the two images always makes the underwater object blurred to human eyes. However, when we observe something under the water surface, why can't we see two virtual images? In addition, we commonly feel the image does not move closer to the observer yet preserves its location directly upwards the real object, creating a diminishing effect on its vertical depth. The answer to this intriguing questions comes from the synthesis effect of our brain. By receiving the signals of two virtual images, our brain will automatically integrate the two images into one, the sagittal image, which is just above the real object.

Even direct observations from our eyes lead to distorted understanding to the true mechanism of this natural world. Human observations foster the development of theoretical frameworks and ideologies, which significantly benefit human beings in comprehending science in a professional way. However, as we confront incomprehensive perceptions to the nature, we are struggled with the same question stated before in this essay: should we believe instinctual observation or theoretical derivation? At this moment, pragmatic experiments will play a pivotal role in revising people's thought process and eliminating the discrepancy between truth and flawed observation. Consequently, the most profound discoveries under this age of complexity do not arise from one-sided inquiries, but from the harmonious interplay of observations, theories, and experiments. Only by being equipped with the standard way of exploring nature can we find out the reasons behind those mysterious and counter-intuitive phenomena of the world.

7 References

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