# Greek salads, food waste, and why tomatoes are most filling in five dimensions (and cucumbers in six)

## Introduction

Picture this - you are a taverna owner in Greece, minding your day, and you are suddenly approached by a couple of 70-dimensional beings, Yianni and Maria, who ask for one simple thing – a Greek salad fit for their 70D stomachs. Now naturally, as they're reasonable people they give you the contacts of a supplier of the necessary ingredients in 70D, and since you're no novice and have quite the number of culinary exploits under your belt, preparing the salad itself will prove no problem for you, but where you're really struggling is knowing how much to order – after all how filling even is a single 70-dimensional vegetable?

Your first order of business is to look through the ingredients and decide how to model the volume of each one, for trying to visualise what an actual tomato shape would look like in 70D is too much even for your prodigious brain. The following is what you decide:

| Ingredient | Model |
| --- | --- |
| Tomatoes | Modelled as a hyperball[1] of radius 1 meter |
| Cucumbers | Modelled as a hypercylinder of sorts, where a cross section of an (n-1)-ball of radius 1 meter is extended 10 meters in the $n^{th}$ dimension. |
| Feta cheese | Modelled as a hypercube of side length 0.5 meters |
| Peppers, Onions, Olives | Modelled as tomatoes |
| Olive oil, Oregano, Salt | Modelled as out of stock |
| One portion | Modelled as 1 cucumber, 10 tomatoes and 10 blocks of feta |

It might also be worth mentioning that volumes will be treated purely numerically, and the differences between a meter to the $n^{th}$ power and a meter to the $(n-1)^{th}$ power will not be explored, so if the numbers derived at the end seem ridiculous, that is in part because they will be.

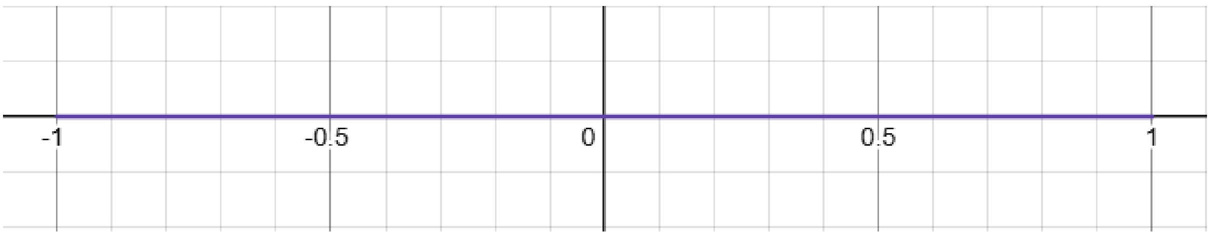
[1] The prefix hyper- generalises things to any number of dimensions, while putting some specific number (or n) in front of an object or concept specifies that exact number of dimensions (e.g. a 3-ball is just a ball, and a 2-ball is a disc)

## A meander through familiar dimensions

Now, since jumping straight to 70 dimensions seems quite the leap, let us start by trying to build an intuition for how volume works in dimensions we can comprehend as well as what our models mean, and then extending that to the fourth and higher dimensions in an attempt to find a more general formula.

Before even starting with the first dimension however, it is helpful to start by defining what is meant by an n-ball, an n-cube, and even an $n^{th}$ dimension. Let's start with the latter – an n dimensional space can be viewed most simply as a place where each point or vector has n different components, where you could imagine n different axes, all mutually perpendicular, that each independently determine one component. An n-cube is then a shape with equal edge lengths, right angles in every dimension, and $2^n$ corners, similarly to how a square works in 2 dimensions or a cube in 3. Fortunately for our strained restauranteur, the area of an n-cube is exceedingly simple to obtain, as it is just the side length to the $n^{th}$ power. Where things get interesting however is the n-ball, which can be defined as the area encompassed by the locus of points at some given distance from the centre, or by the equation $x_1^2 + x_2^2 + x_3^2 + \cdots + x_{n-1}^2 + x_n^2 \leq r^2$, something both easier to deal with analytically, and easier to comprehend intuitively than what equidistance means in dimensions higher than the third. Unfortunately the hypervolume of this is not quite so easy to figure out, so let's embark on an inter-dimensional journey.

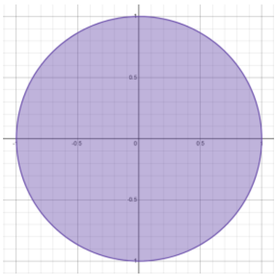
Starting with the first dimension, this consists of just one axis, so our 1-tomato has as its boundary two points, each at distance one meter from some centre (for our purposes the origin), as can be seen below[2]:




Since it's one dimensional, its hypervolume is just a length, 2.

[2] Thanks be to Desmos for its comprehensive and easy to use graphing tools

Moving onto two dimensions, we now have a second axis, and the new equation $x^2 + y^2 \leq 1$, which yields the following:



Now this is a tomato with a much more difficult to calculate hypervolume (also known as area in the two dimensional case). Though we could simply remember the formula for an area of a circle (since that is what this strange shape in fact is), we're looking to glean insight into how we might extend this to dimensions we cannot visualise, so let's try another way. Envision one small sliver of the height of this 2-tomato, taken with some very small width dx such that we can imagine it to be a rectangle. As $x^2 + y^2 \leq 1$, for any x, $y = \sqrt{1 - x^2}$, so the height of this rectangle is twice that and its area is $2\sqrt{1 - x^2}$ times dx, and summing all of those rectangles from the boundaries of the circle $-1 \leq x \leq 1$, we get:

$$\int_{-1}^{1} 2\sqrt{1 - x^2}dx$$

Which we can solve with a substitution of $x = \sin(u)$.

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - sin^2(u)}\cos(u)du$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{cos^2(u)}\cos(u)du$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du$$

And since the antiderivative of $\cos^2(u)$ is $\frac{u}{2} + \frac{\sin(2u)}{4}$ we can evaluate this at the bounds:

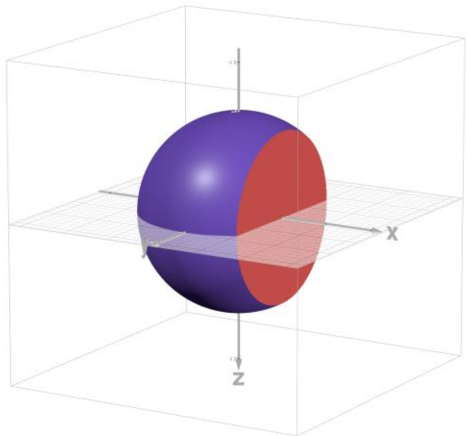
$$2\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + 0 - -\frac{\pi}{2} - 0$$

$$= \pi$$

And after all that we've managed to get an answer that confirms our usual formula for finding the area of a 2-tomato, $\pi r^2$.

Now moving up yet another dimension, we can do much the same thing, except this time we take a 2-tomato shaped slice of our 3-tomato:




Now using the formula for the area of a 2-tomato $\pi r^2$, and the fact that the radius of any such bar will be $\sqrt{1-x^2}$, since this time $x^2 + y^2 + z^2 \leq 1$, but the radius we are looking for is$\sqrt{y^2+z^2}$. Thus the area of any such small disc formed over a small width dx will be $\pi r^2$ times $dx$, which we can sum up between the bounds for x as:

$$\int_{-1}^{1} \pi \sqrt{1-x^2}^2 dx$$

$$= \int_{-1}^{1} \pi - \pi x^2 dx$$

$$= \left[\pi x - \frac{\pi x^3}{3}\right]_{-1}^{1}$$

$$= \pi - \frac{\pi}{3} - -\pi - - - \frac{\pi}{3}$$

$$= \frac{4\pi}{3}$$

Which is, again, the result expected for our 3-tomato based on the pre-existing formula $\frac{4}{3}\pi r^3$.

Here it seems natural to interject that in any number of dimensions, the unit n-ball can be scaled up to an n-ball of any radius by stretching it by factor r parallel to each axis, since each such stretch affects only that one dimension and no other. This means that an n-ball of some radius r has a volume of $r^n V_n(1)$ where $V_n(x)$ is the volume of a ball in n dimensions of radius x.

Now we could continue in this manner, perhaps thinking up some ingenious way of depicting a fourth dimension, with a ball of changing radius through time or some other graphical means, and then repeating this same method of taking the cross section which we've already found a formula for and integrating across the new axis on and on until we get to 70 dimensions. But we wouldn't want to keep Yianni and Maria hungry, so we'll have to think of a more efficient method.

## A formula for the n-tomato

Let's start by taking an n-tomato. Let's look at its cross section when cut by a hyperplane, which will be an (n-1)-tomato of radius $\sqrt{1-x^2}$ for some small width over the n^th dimension $dx$ so we can represent this as:

$$\int_{-1}^{1} V_{n-1}(\sqrt{1-x^2})dx$$

But we've shown that $V_n(x) = x^n V_n(1)$ so we can put this in terms of a unit (n-1)-tomato

$$\int_{-1}^{1} V_{n-1}(1)\sqrt{1-x^2}^{\,n-1} dx$$

$$= V_{n-1}(1)\int_{-1}^{1} \sqrt{1-x^2}^{\,n-1} dx$$

This is an even function due to the $x^2$ so we can change up the bounds a bit.

$$2V_{n-1}(1)\int_{0}^{1} \sqrt{1-x^2}^{\,n-1} dx$$

Now we can substitute in $u = x^2$, so that $dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}$.

$$V_{n-1}(1)\int_{0}^{1} u^{-\frac{1}{2}}\sqrt{1-u}^{\,n-1} du$$

$$= V_{n-1}(1)\int_{0}^{1} u^{-\frac{1}{2}}(1-u)^{\frac{n-1}{2}} du$$

And though here it may come to mind to wonder why anyone would choose to take a complicated integral and seemingly make it worse, we can connect this new integral to the beta function:

$$\mathrm{B}(x,y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1}dt$$

Which is exactly what we have for $\mathrm{B}\left(\frac{1}{2},\frac{n+1}{2}\right)$. And since we've found the Beta function in our solution, a natural next step would be to move over to the Gamma function, and prove the relationship between the two.

Take:

$$\Gamma(\mathrm{x})\Gamma(y)$$

Where $\Gamma(\mathrm{x})$ is the Gamma function:

$$\Gamma(\mathrm{x}) = \int_0^\infty e^{-u}u^{x-1}du$$

So now we have:

$$\int_0^\infty e^{-u}u^{x-1}du\int_0^\infty e^{-v}v^{y-1}dv$$

Which we can bring together since they are two integrals in different variable to get the following:

$$\int_0^\infty\int_0^\infty e^{-(u+v)}u^{x-1}v^{y-1}dudv$$

And at this point, faced with this integral, divine inspiration struck (thanks Wikipedia[3]). Substitute In $u = st$ and $v = s(1-t)$ and we have:

$$\int_0^\infty\int_0^\infty e^{-(st+s-st)}(st)^{x-1}(s(1-t))^{y-1}\left|\frac{\delta(u,v)}{\delta(s,t)}\right|dsdt$$

$$= \int_0^\infty\int_0^\infty e^{-s}s^{x-1}t^{x-1}s^{y-1}(1-t)^{y-1}\left|\frac{\delta u}{\delta s}\frac{\delta v}{\delta t} - \frac{\delta u}{\delta t}\frac{\delta v}{\delta s}\right|dsdt$$

$$= \int_0^\infty\int_0^\infty e^{-s}s^{x+y-2}t^{x-1}(1-t)^{y-1}|(t)(-s)-(s)(1-t)|dsdt$$

$$= \int_0^\infty\int_0^\infty e^{-s}s^{x+y-2}t^{x-1}(1-t)^{y-1}|-s|dsdt$$

But here we can notice that since both $u$ and $v$ range from 0 to infinity, they are positive, meaning either $s, t$ are both positive or both negative. But if both were negative, $1-t$ would have to be negative, which is only possible for positive t which presents a contradiction and thus the absolute value of $-s$ is $s$. Furthermore, since $v$ is positive and $s$ is positive, $1-t$ must be positive so t must range between 0 and 1, so we can change the bounds of one of the integrals:

[3] https://en.wikipedia.org/wiki/Beta_function

$$\int_0^1 \int_0^\infty e^{-s} s^{x+y-2} t^{x-1} (1-t)^{y-1} s\,ds\,dt$$

$$= \int_0^1 \int_0^\infty e^{-s} s^{x+y-1} t^{x-1} (1-t)^{y-1} ds\,dt$$

Here the eagle-eyed mathematician might notice that if we separate out the integrals into one purely with respect to $t$ and one purely with respect to $s$ we get:

$$\int_0^1 t^{x-1} (1-t)^{y-1} dt \int_0^\infty e^{-s} s^{x+y-1}\, ds$$

But this is exactly $\mathrm{B}(x,y)\Gamma(x+y)$. So since what we started with was $\Gamma(\mathrm{x})\Gamma(\mathrm{y})$ we have:

$$\Gamma(\mathrm{x})\Gamma(\mathrm{y}) = \mathrm{B}(x,y)\Gamma(x+y)$$

$$\mathrm{B}(x,y) = \frac{\Gamma(\mathrm{x})\Gamma(\mathrm{y})}{\Gamma(x+y)}$$

So back to our tomatoes, and the $\mathrm{B}\left(\frac{1}{2},\frac{n+1}{2}\right)$ that we derived, we now know this is equal to:

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n+1}{2}\right)}$$

$$= \frac{\left(\int_0^\infty e^{-x} x^{-\frac{1}{2}} dx\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

But this top integral can be evaluated, substituting in $\sqrt{x} = u$ so that $du = \frac{dx}{2\sqrt{x}}$ and:

$$\int_0^\infty e^{-x} x^{-\frac{1}{2}} dx = 2\int_0^\infty e^{-u^2} du$$

But since this is again even, we can instead take it from negative to positive infinity, and get the Gaussian Integral:

$$\int_{-\infty}^\infty e^{-u^2} du$$

And though I won't go through evaluating this, to spare you the time as well as due to a primal fear of what I saw on the Wikipedia page, I can assure you it does in fact evaluate to $\sqrt{\pi}$.

So now if we scroll back up a few pages, we can get to a full recursive formula:

$$V_n(1) = V_{n-1}(1)\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\right)$$

Even better than this, we can now substitute in our formula for $V_{n-1}$ until we hit $V_1$ and we get a neat telescoping sum.

$$V_n(1) = \left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\right)\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right)\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)\dots\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{4}{2}\right)}\right)\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)$$

Here all the terms but two Gamma functions and a bunch of roots of $\pi$ cancel out and we get:

$$V_n(1) = \frac{\Gamma\left(\frac{2}{2}\right)\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)}$$

But since the Gamma function generalises factorials such that $\Gamma(x) = (x-1)!$, $\Gamma(1) = 0! = 1$, so we can simplify further to:

$$V_n(1) = \frac{\Gamma\left(\frac{2}{2}\right)\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$V_n(1) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)}$$

With the added caveat that if for some reason you care about more multidimensional balls than simply our 1 meter radius tomatoes, you can just multiply through by the radius n times:

$$V_n(r) = \frac{r^n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)}$$

## Back to Yianni and Maria

Now that we've found the volume of an n-tomato, lets get back to our hungry couple. We can assume that in the time we've spent calculating this, they've grown appropriately famished and want a meal of 3000 calories each to satiate their hunger. We can also assume that a 70 dimensional food item offers the same energy per $m^{70}$ that a three dimensional one offers per $m^3$. A quick google search reveals that the caloric densities of tomatoes, cucumbers and feta are around 20, 20, 300 per hundred grams respectively, so taking those values and assuming each has about the density of water, we get 200 000, 200 000, 3 000 000 per meter cubed respectively. To calculate the volume of each in turn:

Our 70-tomato has hypervolume $V_{70}(1) = \frac{\pi^{35}}{35!} \approx 1 * 10^{-23} m^{70}$, so about $5 * 10^{-18}$ calories per 70-tomato.

Our 70-cucumber has hypervolume $10V_{69}(1) = \frac{10\pi^{\frac{69}{2}}}{\Gamma\left(\frac{71}{2}\right)}$. Now this $\Gamma\left(\frac{71}{2}\right)$ may seem a bit scary, but one feature of the Gamma function is that $\Gamma(x+1) = x\Gamma(x)$, and we know $\Gamma\left(\frac{1}{2}\right)$ so we can set up a recursion:

$$\Gamma\left(\frac{71}{2}\right) = \left(\frac{69}{2}\right)\left(\frac{67}{2}\right)\left(\frac{65}{2}\right)\dots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \left(\frac{69}{2}\right)\left(\frac{67}{2}\right)\left(\frac{65}{2}\right)\dots\left(\frac{1}{2}\right)\pi^{\frac{1}{2}}$$

$$= \frac{69!!\,\pi^{\frac{1}{2}}}{2^{35}}$$

Where '!!' is the double factorial function, multiplying together all odd integers from 1 to 69, and so our final volume is:

$$\frac{10\pi^{\frac{69}{2}}}{\left(\frac{69!!\,\pi^{\frac{1}{2}}}{2^{35}}\right)}$$

$$= \frac{10\pi^{34}2^{35}}{69!!} \approx 8 * 10^{-22} m^{70}$$

So about $1 * 10^{-16}$ calories per 70-cucumber.

The 70-feta is the easiest of the bunch, as all we need to calculate it $\frac{1}{2}^{70} \approx 8 * 10^{-22}$ so about $3 * 10^{-15}$ calories per cube of 70-feta.

## Food waste

But before that, we have a problem, since as any seasoned restauranteur, you can't just trust some unknown supplier – maybe they've used some nasty 70 dimensional pesticides, or unclean production methods, so the vegetables must be peeled.

Some in depth study of the picture above reveals that we can simply model the area needing to be peeled as the outer $\frac{1}{10}$ of the radius, for a particularly untrustworthy supplier. So let's see what this does to our vegetables.

For the tomato, we simply take $V_{70}\left(\frac{9}{10}\right) = \frac{\left(\frac{9}{10}\right)^{70}\pi^{35}}{35!} \approx 5 * 10^{-27}$ so about $1 * 10^{-21}$ calories. More than a thousand times off, just by taking 10% of the radius away. This may not be very intuitive – after all looking at 2 and 3 dimensional balls, it's not really true that most of their volume is near their ends, but if we look at the rate of change of our 70-ball's hypervolume, it's proportional to its radius to the 69th power, so it's only natural that it grows at such a fast rate that even the outer hundredth of it still constitutes a majority of volume.

Doing this same calculation with our cucumber, we get about $6 * 10^{-20}$ calories, and since fortunately we have no need to cut down on feta (though one could question where the volume of a hypercube is concentrated[4]), we can now find the caloric content of our portion:

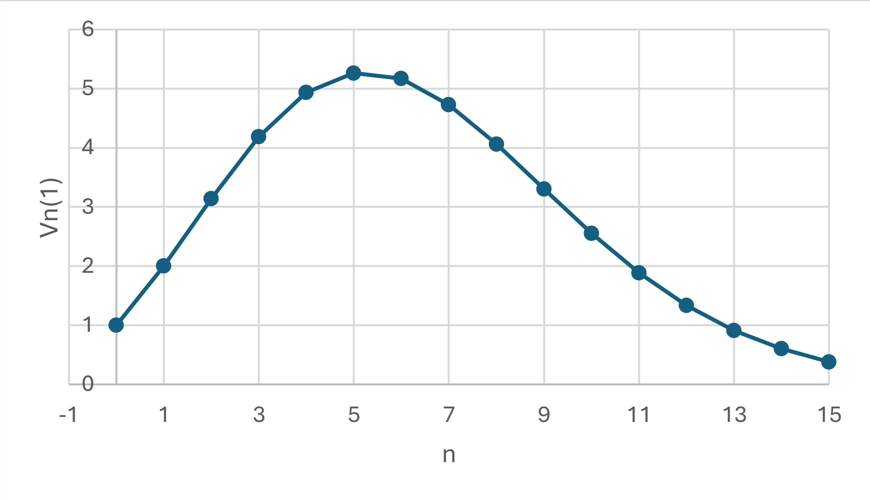
$$1(6 * 10^{-20}) + 10(1 * 10^{-21}) + 10(3 * 10^{-15}) \approx 3 * 10^{-14}$$

So to feed their required 6000 calories, we need a grand total of $2 * 10^{17}$ portions of salad.

[4] For an answer to this, imagine an n-ball in the centre of an n-cube, and keep increasing its radius until it touches the walls of the cube, and then maybe beyond, while calculating the percentage of the volume of the n-cube that the n-ball occupies.

## The best dimension?

And now, after Yianni and Maria have left, happy with their meal, and the bill has come in for quadrillions of vegetables, you ask yourself 'why couldn't they have been of a different dimension?'. And in fact, what dimension would be the cheapest (assuming constant prices per vegetable)?. Well the feta is the easiest yet again, as every time you go up a dimension, its hypervolume halves, so the $0^{th}$ dimension where it has a measure of 1 is the most effective. The cucumber and tomato however need a bit more work, but by simply graphing the hypervolume of an n-ball you get the answer pretty clearly:




So the highest volume of an n-tomato is found in five dimensions, and similarly an n-cucumber in 6. And thus you now know for future such mishaps that higher dimensional orders are only worth taking if you're approached by a 0 dimensional being, a 5 dimensional being and a 6 dimensional being asking to share a Greek salad.

## Conclusion

So does this mean anything? Not really - if we go back to the start, where I said I'd be ignoring units for the sake of my salad, that's actually quite a big thing to ignore. Had we instead taken vegetables of radius 0.1 metre, our optimal vegetables would have changed dramatically in dimension and, even more to the point, if we had taken vegetables of radius 100cm, their volume even in 70 dimensions would have been measured in hugely positive powers of 10, rather than what we ended up with.

This is because when cropping out the physical meaning of the numbers, and treating them purely algebraically, we forget that as nice as it is to do all the fun maths, if I made someone a bunch of red circles and green rectangles, they wouldn't be happy with their salad even if I reassured them that it did in fact have the same numerical measure as an actual salad would have.

So is this formula we've derived useless? Not at all. Apart from its uses in various scientific fields, it's important to remember that an n-ball really is just n numbers the squares of which sum to less than or equal to its radius squared, and even in a two or three dimensional setting, this is by no means a rare situation[5].

[5] One fascinating example of a two dimensional puzzle which uses this formula to come to a solution can be found here: https://www.youtube.com/watch?v=6_yU9eJ0NxA&t=10s&pp=ygUnbnVtYmVycGhpbGUgZGFydGJvYXJkIGhpZ2hlciBkaW1lbnNpb25z