Hair Maths: Curl Geometry and Braids

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1.1 Introduction

I struggled like crazy when trying to decide what to write about for this essay. Finding the right balance between entertainment, originality and educational content felt like a gargantuan task. I wanted to present something most people probably hadn't considered could be related to maths. I asked my dad for help with ideas (there's the credit you insisted that you deserve for this essay, dad), and he told me to just 'write about what I know and like'. The stress of coming up with an idea caused me to sit there, silently, for about 10 minutes straight, just twirling my hair around my finger – a nervous habit. I found myself thinking that the curl was just such an interesting shape, so I wrote about that. I hope you enjoy!

1.2 Unfamiliar Variables

Curl Geometry

Number of Turns = n

Total Helical Length = l

Pitch Helical Length = $\frac{l}{n}$

Total Apparent Length = h

Pitch Apparent Length = p

Shrinkage Constant = k

Curliness = c

Braids

Identity Braid = I

Intersections = a, b, a^{-1}, b^{-1}

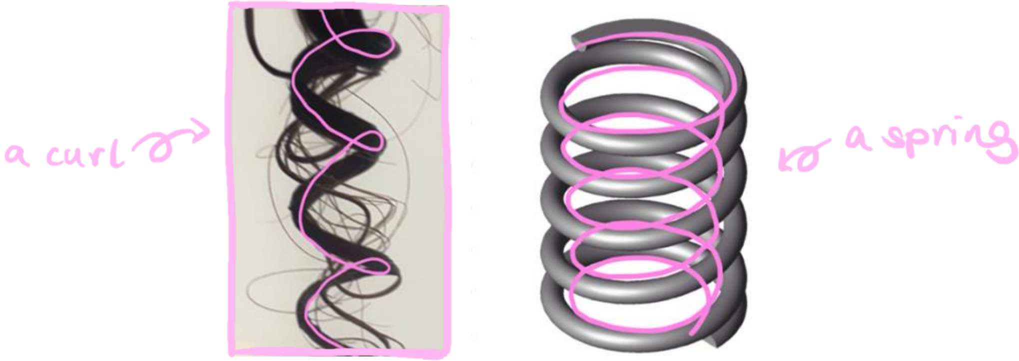
Number of Braid Units = n

2 Helical Hair

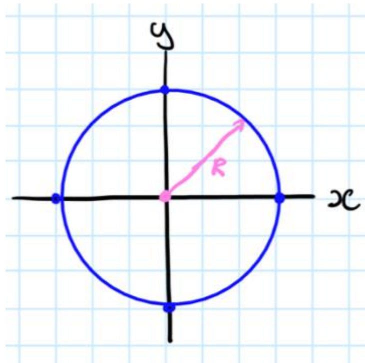
2.1 Modelling a Curl

Curly hair: a phenomenon which fascinates those without it, and endlessly frustrates those with it (coming from a lifelong sufferer), but how do we model the shape that creates this mesmerising pattern?

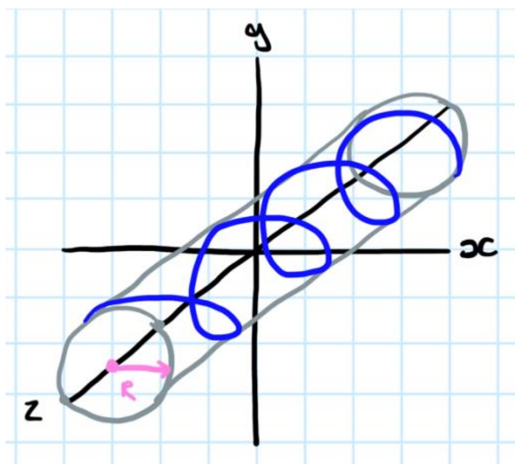
If each curl is considered individually, it is clear that their shape resembles that of a spring:



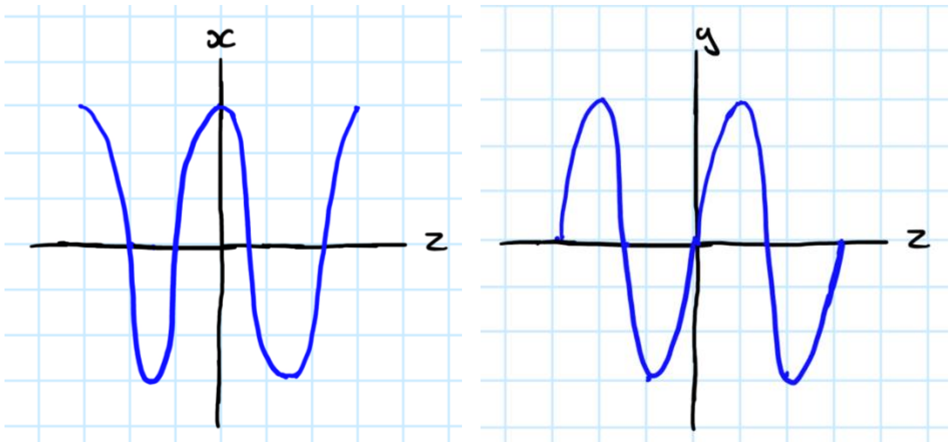
The name for the shape made by a spring is a 'helix'. Think of a helix as though it is a length of string wrapped around a cylinder; this implies that when viewed from above, the shape seen is our old friend — the circle.



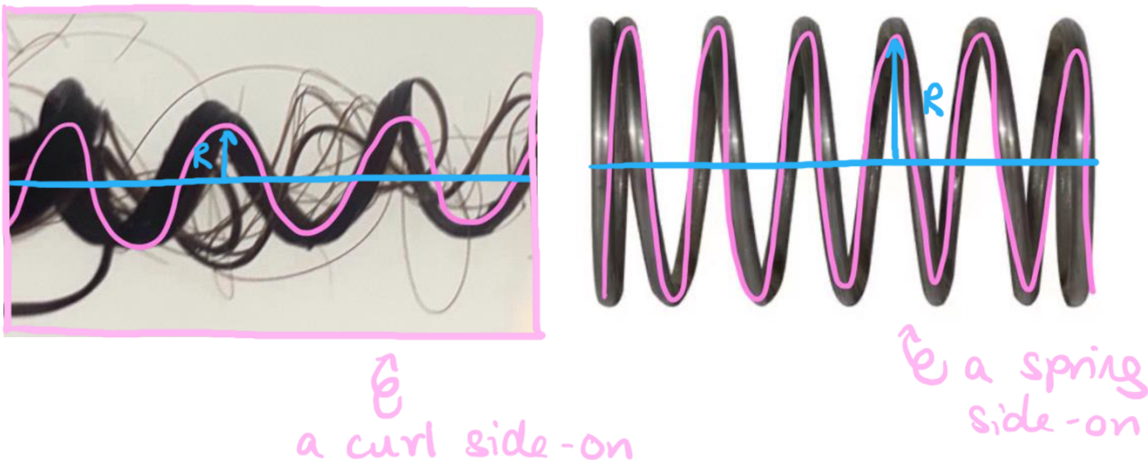
The shape seems more approachable now, right? Assuming the centre of the circle is (0,0), we can assert that the equation for the shape formed by the *xy* graph is $x^2 + y^2 = R^2$, where R is the radius of the circle. If we label the radius of the circle and return to our previous 3D image of the helix (where the *z* axis represents depth), we can see that the radius of the circle is the same as the radius of the helix.



Interestingly, the 2D graphs formed with the *zx* and *zy* axes are also familiar to us — they form a cos and sin curve respectively (trigonometry really is everywhere)!



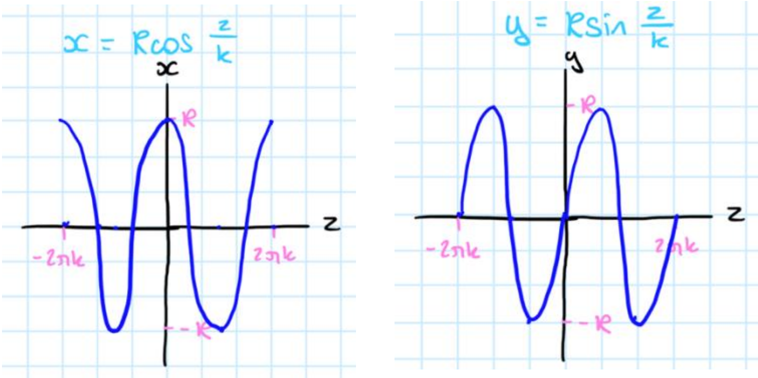
You can see that a helix will make these shapes in the *zx* and *zy* axes by looking at a helix side-on:



Now, how do we create a set of equations to model a helix? Well, if we consider the parameter '*t*' to be the point of time in a particle's constant motion about the helix, we can then assert that $z = kt$, where k is a the constant speed of the particle and $k \neq 0$, and z is the distance travelled across the z axis. $v = \frac{s}{t}$, so $s = vt$ and therefore $z = kt$.

The *xt* and *yt* graphs can be modelled as $x = R\cos(t)$ and $y = R\sin(t)$, with R acting as a coefficient because the maximum displacement from equilibrium has an equal magnitude to the radius of the helix.

These equations can be rewritten by rearranging $z = kt$ as $t = \frac{z}{k}$. You can then substitute this into the equations for x and y, rewriting them as $x = R\cos(\frac{z}{k})$ and $y = R\sin(\frac{z}{k})$.

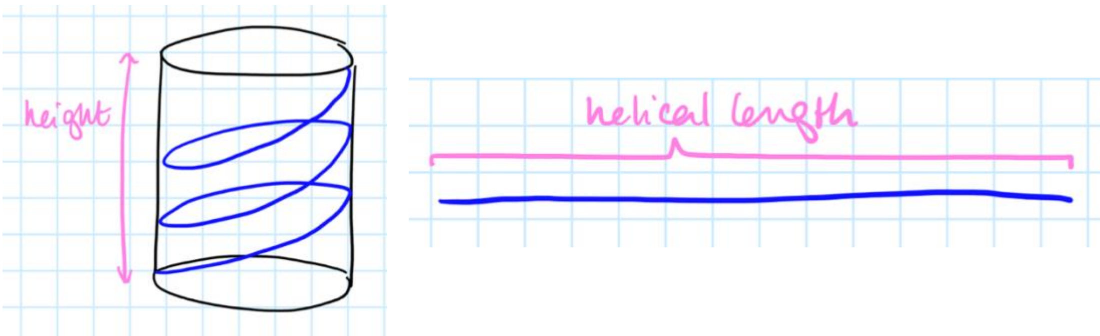


And so we have achieved a set of equations for a helix, which can be applied to a curl:

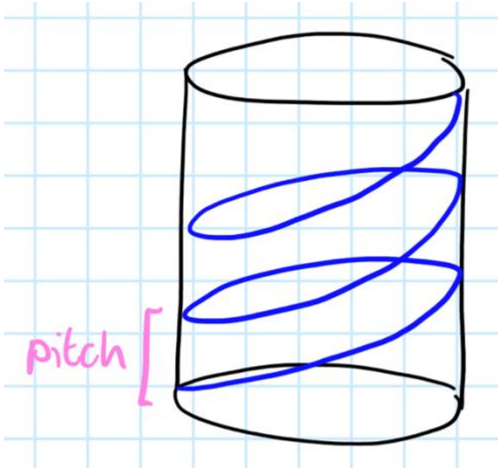
$$z = kt,\ x = R\cos(\frac{z}{k}),\ \text{and}\ y = R\sin(\frac{z}{k})$$

2.2 How long would my hair be if it was straight?

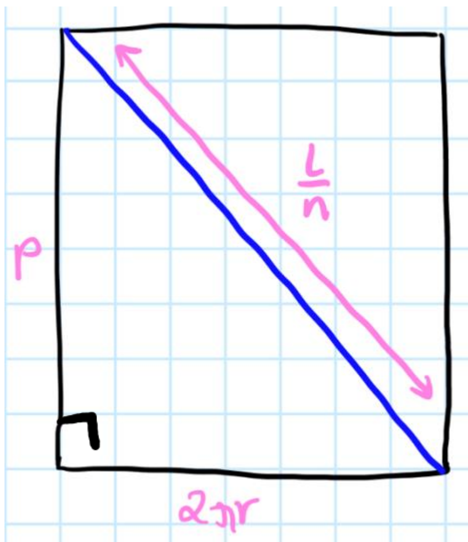
My hair takes ages to grow. This is not due to some genetic defect (I hope), but rather the simple fact that I have curly hair, which results in something that we in the curly community refer to as 'shrinkage', which is when your hair appears shorter than it actually is due to its curls. This can be mathematically explained using helices; the 'helical length' of a helix is longer than it's height, where the term 'helical length' refers to the actual length of the line, or in this case the strand of hair making up the curl.



The 'pitch' of a helix is the height of one complete 'turn', which is basically one loop around our imaginary cylinder.



Now, imagine you have a helix with just one pitch. If you were to unroll the imaginary cylinder, it would form a rectangle with a base equal to the circumference of the cylinder, a height equal to the pitch of the cylinder, and a diagonal length equal to the helical length. Using the pythagorean theorem ($a^2 + b^2 = c^2$), we can now find an equation for the helical length of one pitch of a helix ($\frac{l}{n}$):



$$\frac{l}{n} = \sqrt{}(p^2 + (2\pi R)^2)$$

Armed with our newfound equation, we can return to our original helix. We should count the number of turns there are around the helix (n). For each of these turns, the helical length of that individual turn is equal to the helical length of one pitch, therefore the total helical length of the helix (l) can be found using:

$$l = n\sqrt{}(p^2 + (2\pi R)^2))$$

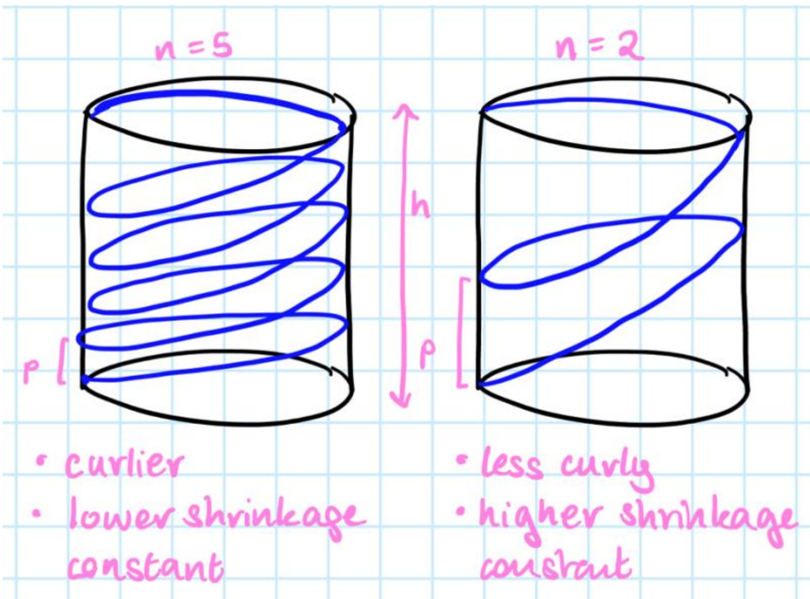
Which is simply a rearrangement of our earlier equation for the helical length of one pitch. So how does this relate to shrinkage? Well, let us assume that a completely straight section of hair can be defined as having 0 shrinkage. This would mean that the apparent length (h) of hair is equal to the helical length (l) of hair, so $\frac{h}{l} = 1$ for straight hair. I'm going to refer to this value as the 'shrinkage constant' (k) (inspired by the spring constant — thank you Hooke's law!) for the remainder of this essay.

$$k = \frac{h}{l}$$

Let us define 'curliness' (c) as being the number of turns of the curl (n) per unit apparent length (h), so $c = \frac{n}{h}$. For a person with completely straight hair, $n = 0$, therefore $c = 0$.

$$c = \frac{n}{h}$$

The more turns per unit length a persons hair has, the curlier their hair is. The curlier someone's hair is, the more shrinkage they will experience, so the shorter their apparent hair length, and the smaller their shrinkage constant.



I completely made up the variables c and k, so I decided to put these equations to the test using my own hair: I measured the apparent length and diameter of 5 curls and counted the number of turns for each curl. I then stretched out the curl to measure its actual helical length to compare with the calculated value I would obtain. I then found a mean for my own hair's curliness and shrinkage constant. Here is a table showing the results I obtained:

Refer to 1.2 for a reminder on variables

= measured   = calculated

	h (cm)	d (cm)	n	l (cm)	L (cm)	p (cm)	k	c
1	22	0.9	15	49	48	1.5	0.45	0.68
2	28	0.9	17	56	55	1.6	0.5	0.61
3	19	1	12	46	42	1.6	0.41	0.63
4	20	0.8	14	48	40	1.4	0.42	0.7
5	17	1	12	44	41	1.4	0.39	0.71
						mean	0.43	0.67

equations used:

* $p = \frac{h}{n}$
* $k = \frac{h}{l}$
* $c = \frac{n}{h}$
* $L = n\sqrt{p^2 + (2\pi r)^2}$

So, my hair has a shrinkage constant of 0.43, and a curliness of 0.67! This means that my hair appears 2.3x shorter than it *actually* is ($2.3 = \frac{1}{0.67} = \frac{1}{k}$), so now at least I can provide a more accurate answer to the question 'So how long is your hair when you straighten it?' (a surprisingly common question from the $c \approx 0$ subsection of the population). I think overall the calculations were pretty accurate, which you can see by comparing the actual and calculated values for the helical length of my curls. The accuracy of course varies, as this is just a model, and a model will never be perfect, but I'm still happy!

3 Braids

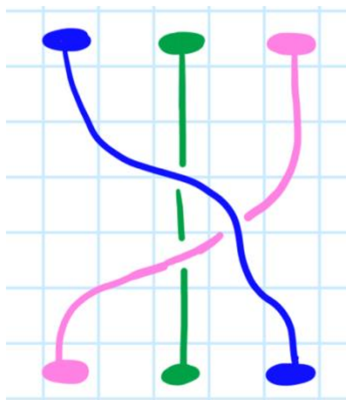
3.1 Mathematically Representing Braids

Honestly, although I love my curls (even more now I understand the maths hidden in them), I don't actually wear them down that much. They take forever to style, and get in the way all day anyway. High effort for a low reward isn't the greatest trade-off. So, I decided to also talk about the maths of a hairstyle which I tend to wear out and about more often: braids.

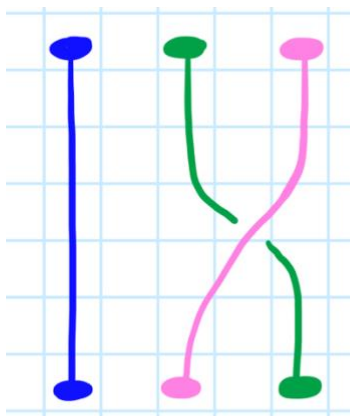
I've always loved braided styles. There's something so intrinsically mathematical about the patterns formed by the interwoven strands of hair, which I noticed and was fascinated by long before looking into it for this essay – so I'm thankful for the excuse to learn more about the topic.

I will be writing about 3-strand braids, as those are what I typically do when I braid my hair, however you can explore braids with literally any number of strands, not just 3.

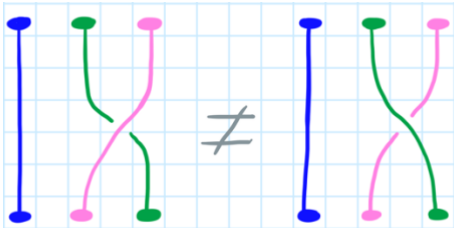
So, let's think of a braid as n strands attached at fixed points on 2 parallel planes. The interesting part of the braid is the path they take to get there, and the intersections of the strands, because we consider these strands to be physical objects, meaning they can't pass through each other.



The simplest braid we can consider (aside from one where each strand just moves straight down), would be one with only one intersection:

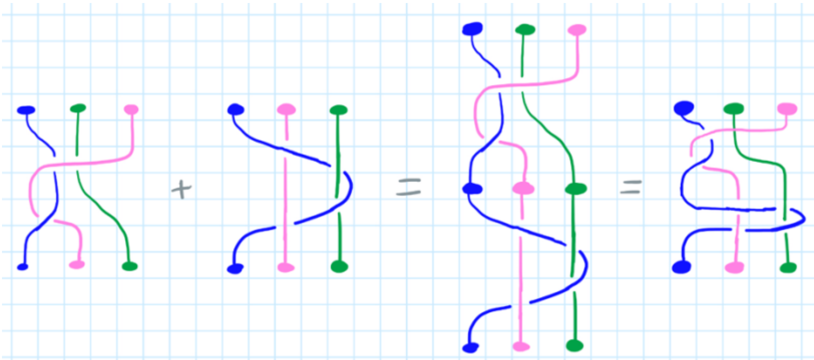


Simple braids like this one are the building blocks for the standard French and Dutch braids that we usually braid into our hair, which I will talk about more later on. It is important to note that it *does* matter which strand is going over which:



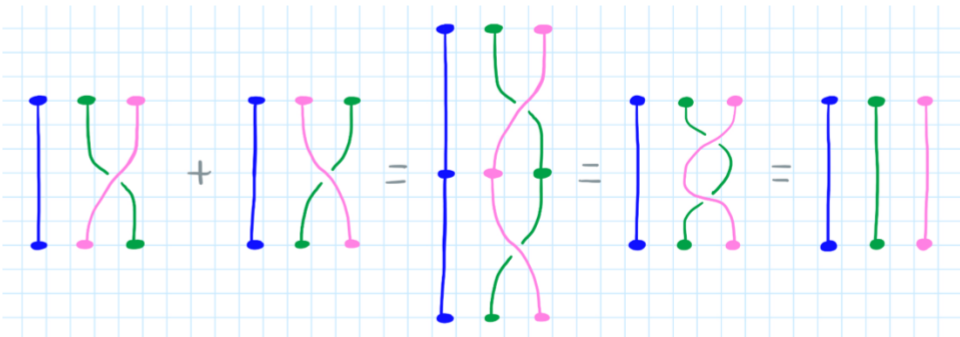
In fact, these 2 braids specifically are not only inequivalent, but are actually the inverse of each other!

Braids can also be added together, the process of adding 2 braids together looks like this:



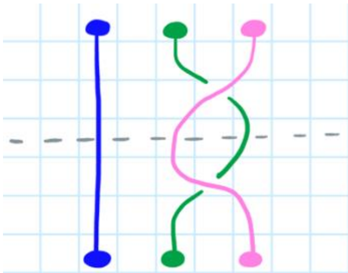
You basically just place the first braid on top of the second one, and then compress it down to size. It is also important to note that (in most cases) the order in which you add together the braids matters – the resultant braid is likely to be different depending on the order.

When you add together 2 inverse braids, the result is always the identity braid, which is the braid that results in no change when added to any braid. Here is an example using the simple braid already shown and its inverse:

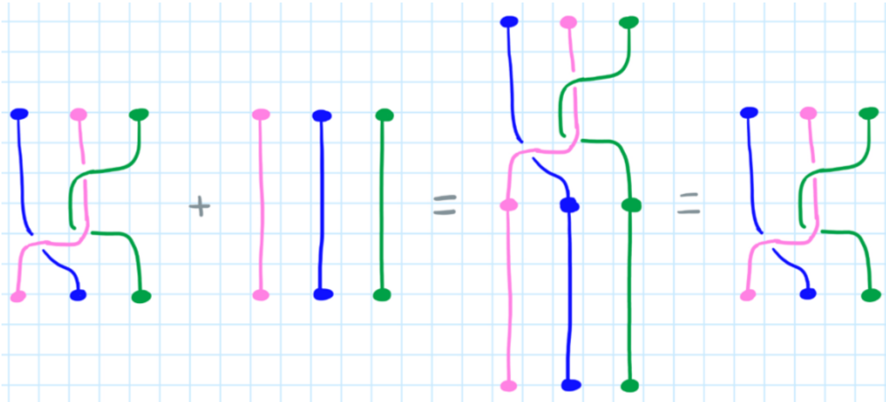


This is possible because braids are moldable. If you think about an actual set of strings in real life, you would be able to move them over each other, so the strands of the resultant braid are able to be moved over each other to form our identity braid. This is possible as long as the strands do not have to be moved through each other, because as we know, we can't move solid physical objects through one another!

You can tell by looking at it that a resultant braid is actually the identity braid because the braid will have a horizontal line of symmetry through the middle:



So, what actually *is* the identity braid? Think of the identity braid like the number 1 in multiplication; any number multiplied by 1 results in itself, and any braid added to the identity braid results in itself:



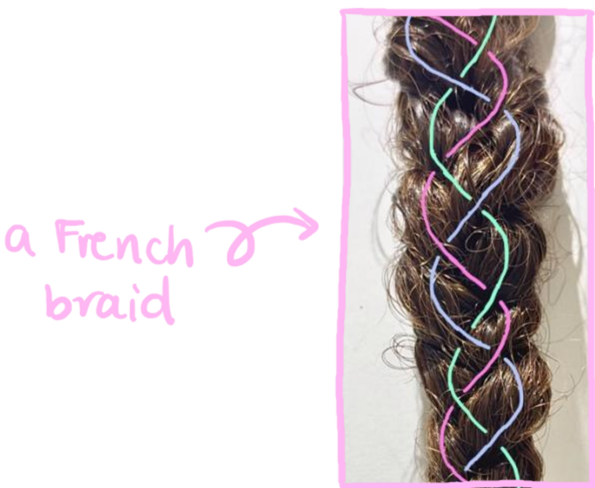
With this in mind, let's from now on denote the identity braid with 'I'.

And those are the basics of representing braids!

3.2 French and Dutch Braids

Now that we know generally how to look at braids mathematically, let us try specifically modelling French and Dutch braids.

So, a French braid is a braid where you take 3 strands of hair and repeatedly cross the outside strands of hair over the middle strand, alternating sides each time.

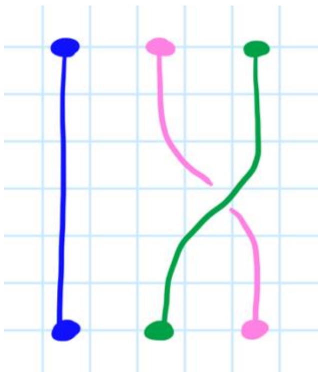


A Dutch braid also involves taking 3 strands of hair, however you instead repeatedly cross the middle strand of hair over the outside strands, alternating sides each time.

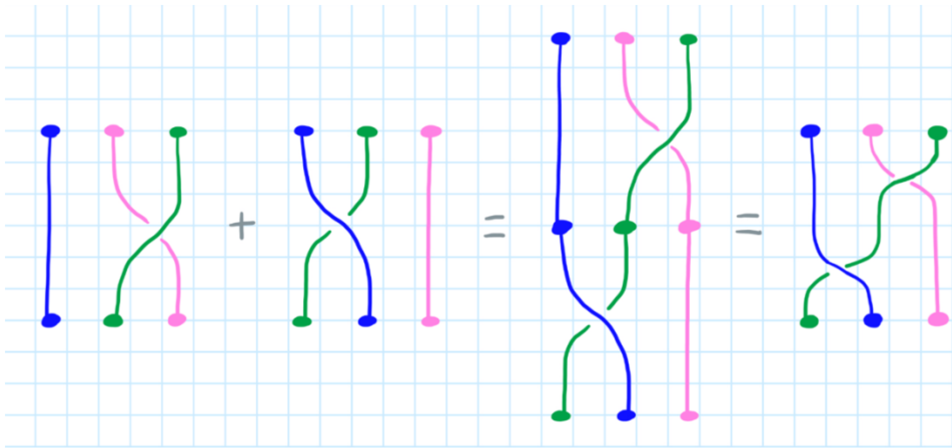
These are the two styles of braids that most people – including myself – are most familiar with.

Let's take a look at how we would represent these 2 braids using braid theory, starting with the French braid. We can initially consider each step of the braiding process as an individual braid and add them together, as opposed to considering the braid as a whole:

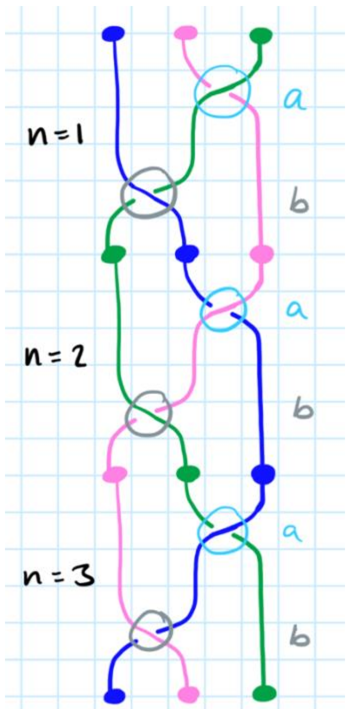
1. Cross the right strand over the middle strand:



2. Cross the left strand over the middle strand:



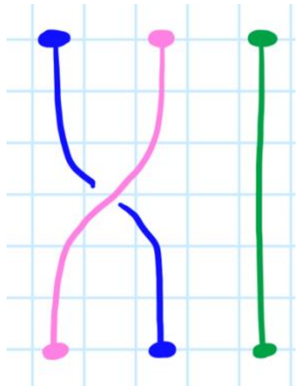
3. Repeat! If we call the intersection of the right strand over the middle strand *a* and the intersection of the left strand over the middle strand *b* we can rewrite our French braids symbolically:



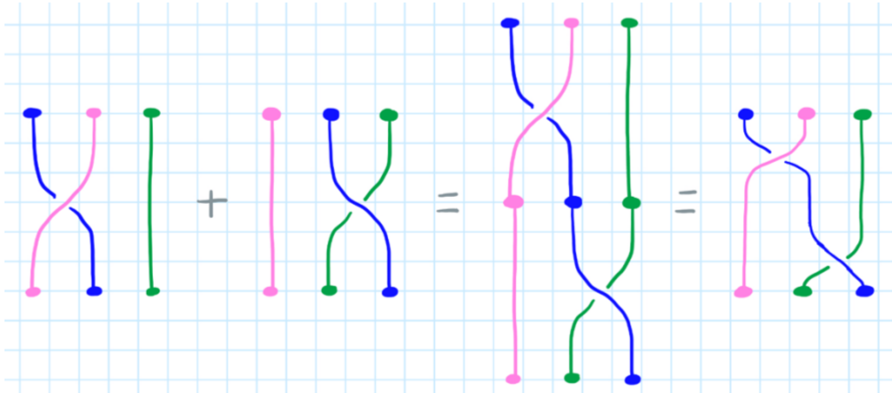
This means our French braid can be written as $(ab)^n$, where *n* is the number of repeats of steps 1 and 2.

Now, let's take a look at the Dutch braid:

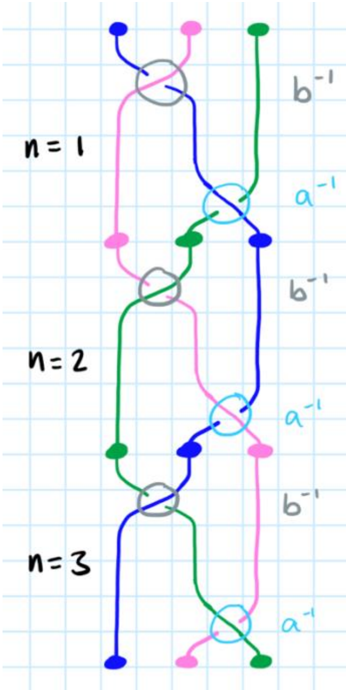
1. Cross the middle strand over the left strand (in practice it doesn't really matter which side you start with, but this method will provide a more interesting result later on):



2. Cross the middle strand over the right strand:

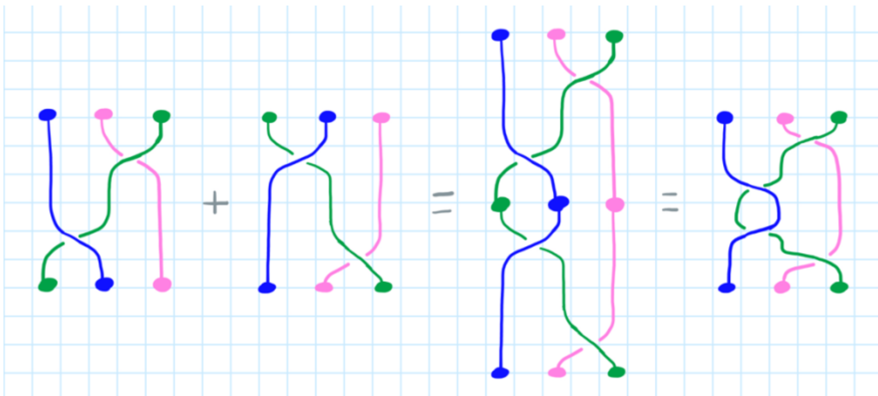


3. Repeat! We can call the intersection of the middle strand over the right strand a^{-1} and the intersection of the middle strand over the left strand b^{-1}. These intersections can be written as the inverses of a and b respectively as they will cancel them out to form the identity braid. Think of this again like with multiplication: $xx^{-1} = 1$. Likewise with braids, $aa^{-1} = I$. We can now rewrite our Dutch braid symbolically:



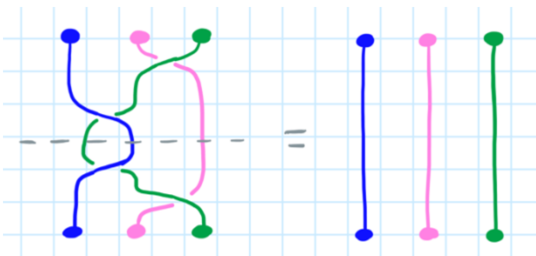
This means our Dutch braid can be written as $(b^{-1}a^{-1})^n$.

So, what does it look like if we add the repeating unit of our French and Dutch braids together?



Well, that was boring.

Wait a second… there's a line of symmetry! That means that the resultant braid is the identity braid!



As we know, when you add together 2 braids and the resultant braid is the identity braid, this means that the 2 braids are each others inverse. The repeat unit of French braids is the inverse of the repeat unit of Dutch braids, and vice versa!

This makes logical sense if we take a look at our symbolic representations of the braids:

French braid unit + Dutch braid unit

$$= abb^{-1}a^{-1}$$

$$= a(I)a^{-1}$$

$$= aa^{-1}$$

$$= I$$

Isn't that interesting? Or at least, I think it is – two of the most popular braiding styles are the mathematical inverse of each other!

4 Conclusion

I know that the maths I presented in this essay may not be particularly complicated, but I'm a firm believer that maths doesn't have to be complicated to be interesting. I went into this essay with the mindset that I really wanted to write something that would allow any reader with GCSE-level mathematical knowledge to learn something new (without getting too much of a headache). I really hope my desire to do that, as well as to share some knowledge of the maths hidden in a topic not often seen as mathematical – but still incredibly prevalent in so many people's lives – has come through. Thank you so much for reading!

5 References

All diagrams drawn by me, all photos of hair taken by me

https://youtu.be/RJ-G9DFbcLA?feature=shared

https://youtu.be/Ce64D_yGDvY?feature=shared

https://youtu.be/qnmTcE5-HpI?feature=shared

https://youtu.be/G_uybVKBacI?feature=shared

https://ocw.mit.edu/ans7870/18/18.013a/textbook/HTML/chapter15/section01.html#:~:text=Here's%20an%20example%3A%20x%20%3D%20cos,of%20some%20particle%20in%20time.

https://seattlemathmuseum.org/math-in-real-life/braids