Why our Democracy is mathematically impossible (and how to fix it)

1.1 Introduction

Democracy is the cornerstone of modern politics. It is seen as the fairest form of election, in which every person has a chance to express their opinions and contribute to a larger group decision. Some claim the ancient Greeks were the first to adopt a valid form of democratic election, which we use whenever we want to elect a leader. But how fair is it? John Addams said, “There never was a democracy yet that did not commit suicide”, showing the fragility and lack of accurate representation in some cases.

In this essay, I want to touch on Kenneth Arrows's social theory work, which earned him the Nobel Prize in 1972, proving how democracy with our current system is never truly possible, as there will either be a group underrepresented or overrepresented. But why do we have to use the same version of democracy that we have used for thousands of years? The quote “If it isn't broken, don't fix it” perhaps doesn't apply here because who says our democracy isn’t broken?

This essay isn't about political opinions or extremist views but about how we elect our leaders differently to represent the largest possible portion of the population.

2.1 First Past the Post

First-past-the-post has been a voting system in Britain since the Middle Ages. It has expanded to all other British colonies and is now used in most US states to host elections.
Its concept is simple: The party with the most votes wins. This would seem trivial and fair, but it has many problems.
Take Scenario A; three parties are {A, B, C}. A and B are very similar in views, so anyone who votes for A wouldn't be unhappy if B were elected and vice versa. C is very opposite in views.
When it comes to the elections, the results are as follows:
A - 25% B - 35% C - 40%.
With our method of voting C wins, however it's only 40% of the population left happy (leading an overall minority) However if B or A won, that would lead an overall 60% happy, because of the unbiased between voters of B and A. Therefore because of the overall percentage of people happy with the result, can we say this is the fairest voting system?

2.2 Multi-round system

One possible way to get around this is with re-elections. This system removes the party with the lowest number of votes, and a re-election is held with one less party. However, for the sake of time and expenses, we could just have everyone rank the parties from their favourite to least on the original vote. This is statistically the same as a re-election, as when one party is eliminated, the re-evaluated vote is just used as the second choice of the people who voted for the eliminated party. This solves our original problem.
Because A received the fewest votes and A and B hold similar views, most people who voted for A would have B as second. Hence, in the second round of the vote, B would receive 60%,

with C receiving 40%. This is a fair conclusion. However, the ranking system does introduce some significant problems.

2.3 The Issues

Issue number one with this system goes as follows:
Repeat 3 Parties, { A, B, C,} but this time, A has extreme views in one direction, C in the opposite direction, and B in the middle.
If we take Scenario B:
A-27% B-30% C-43%
Using our voting system, we see that A is eliminated, and we know A and C hold no view so that the overwhelming majority will hold B as second. Hence, the second round will appear as:
B - 57% C-43%
Leaving B the winner.

However, take Scenario C:
In this scenario, C has a horrible speech before the election, and some of their voters completely flip to voting A or to B
This leaves the election like so:
A-33% B-32% C-35%
In this scenario, B has the fewest votes, and because they are central, it's 50/50 between A and C regarding who was put second.
Hence, the second round appears as:
A-49% C-51%
Leaving C the winner. As we can see from these two scenarios, C doing worse in the first round of the first election led them to win.

The other problem is dubbed ‘The voting paradox’, and it goes as follows:
Scenario D:
Let there be 3 parties {A,B,C} and 7 Voters {V1,V2,V3}:
V1 votes as follows: 1. A   2. B   3. C
V2 votes as follows: 1. B   2 .C   3. A
V3 votes as follows: 1. C   2. A   3. B

Without multiple rounds, we can see a problem. Whatever we proclaim here can be taken as incorrect. Let's say Aiss. Upon further observation, we can see that C is the winner by two-thirds of the time. Therefore, C is the winner.
However, 2/3 times B>, C, so we can assume B is the winner.
However, 2/3 times A>B, so is A the winner?
This leads to a loop and a contradiction.

3.1 Arrow's axioms for democracy

The first is that a system should have an unrestricted domain. If data is inputted into it, it must make a consistent decision and not be uncertain.

The second is a non-dictatorship, which states that one person can not decide the outcome of a vote; an entire group must control it.

The third is non-imposition. This states that if every unit prefers A>, B then the result should proclaim A > B

The fourth is independence. A third option, C, shouldn't affect that outcome when deciding between A or B. A>B shouldn't change to B>A if voters change their preference of A>C
The final is ordinal preference. If A>B and B>C, we can also assume that A>C.

These axioms for democracy are the laws for basic ranking systems. If all five hold in every scenario, we have a true democracy. However, as Kenneth Arrow proved in 1972, this can't happen. So, how can we make it more fair?

4.1 Rated voting systems

A rated voting system goes as follows: instead of the polarising ranking of candidates, or the binary yes or no of candidates. To every candidate, you get them a ranking, out of ten. Then, once all results are in, we would carry out a mean on the data to find a proportional average representing the state's overall opinion.

This form of voting makes it more dynamic. By implementation, voting isn't as simple as A>B>C; it resembles a vector quantity; A is x greater than B. It would also solve some issues with unrepresentative results. I pose this question: If a party gets placed second on every election sheet, is it fair to say that party is the most disliked?

Well, with the previous forms of voting mentioned, it would be instantly eliminated because it held 0% of the popular vote.

4.2 So what would this look like?

To visualise this, we are going to use a simplified model.
In this model, there will be three Parties and three voting templates. Each template favors one of the parties and provides a percentage of when each will occur.

We will assume a scenario similar to our first example. A (Red) is intense in one direction, B (Blue) is strong in the other direction, and C (Green) is very neutral.

A table representing this data would look as follows:

	Party A	Party B	Party C
Voters A = x%	8/10	d/10	2/10

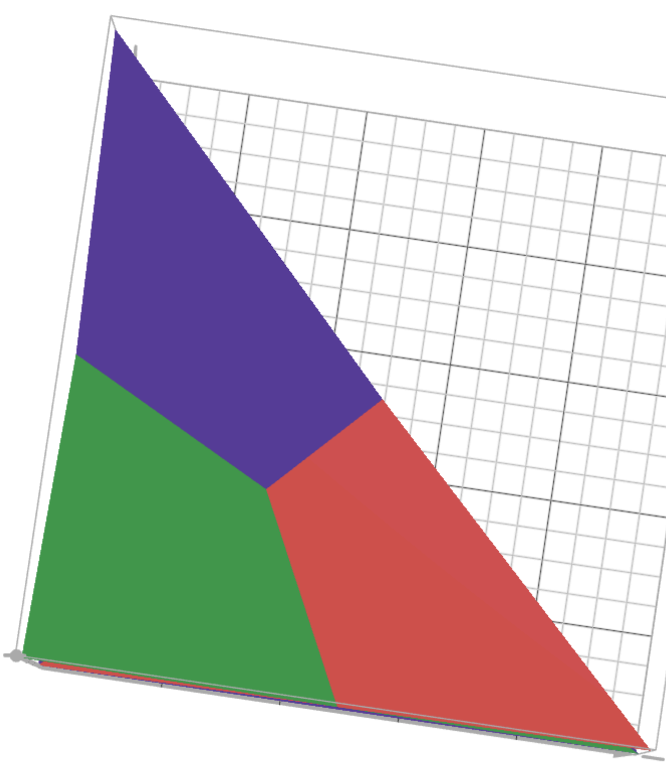
Voters B= y%	2/10	d/10	8/10
Voters C= z%	d/10	8/10	d/10

We will change d/10, which shows the intensity of the midpoint of voting, depending on the people's judgment of the middle party.
We can assume 2>d>8; otherwise, it wouldn't be a midpoint.

We also know that z% = 1 - x% - y,% which will be important in a moment.

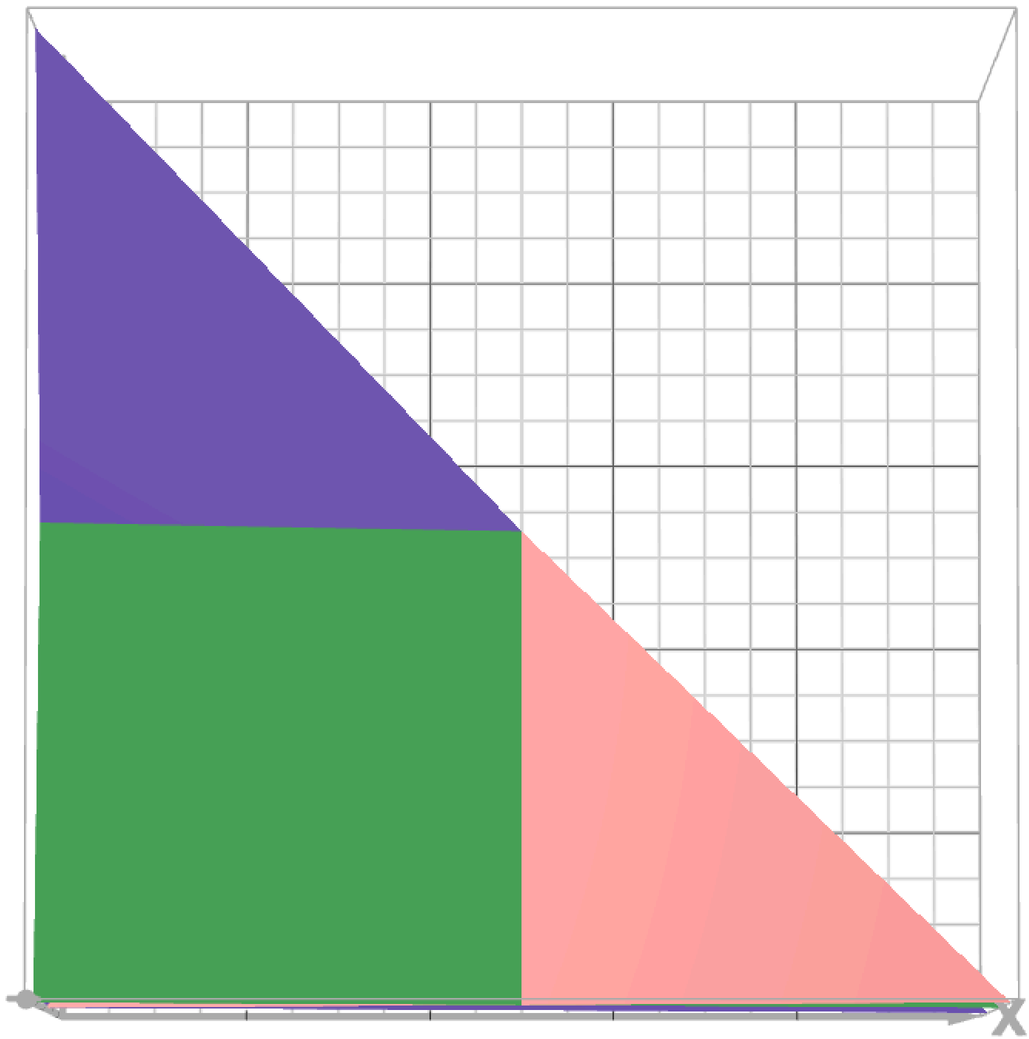
The mean of party: A = 8x/10+2y/10+d(1-x-y)/10
B = dx/10 +dy/10 +8(1-x-y)/10
C = 2x/10+8y/10+d(1-x-y)/10

We can see that A, B, and C are all functions of x and y, with d as a constant. Hence, we can plot these means against each x and y, the percentage of votes party A and B acquired, to see which party is the most outstanding in this scenario.
Plotting all the means on an axis with A, B, and C on the z-axis and looking from the top, we see this graph.



(where d = 2)

If we view this resultant plot from the top, the green area is the percentage where party C wins, blue is where party B wins, and red is where party A wins. At d = 2, we see three equal areas, suggesting that all three parties have an equal chance of winning. However, we see something different at d = 5 as the midpoint.



We see the green region dominate, and with no excess maths, we can see that the total area of the birds-eye result is ½ Green ¼ Blue and ¼ Red. The only way for A or B to win is if they get over 50% of people favoring them heavily.

Even more shocking is to take point (0.45,0.45,z). In this vote scenario, 45% were Voter A (favoring party A), 45% were Voter B (favoring party B), and only 10% put C, the central party, first. However, because it is the second option for the other 90%, it is still the favorite in this system.

4.3 But what's the probability of my party winning at any given value of d?

Following this link, https://www.desmos.com/3d/m8cml9ixnr (demos 3D Graphing calculator), we see that all three graphs for the means are sloping, ‘flat’ angled planes, which appear triangular due to our bounds.

The red plane (Voter A) will be referred to as plane A, the blue plane (Voter B) will be plane B, and plane C is the green plane.

To calculate the probability of a party winning for any value of x% of the vote for A and y% for B, we need to calculate the party's top-down area (ignoring Z) and multiply it by 2.

When d<= 5, we can see the top view of the green area creates a quadrilateral that can be split into two identical triangles. We need to calculate the coordinates of all three vertices to calculate the coordinates. We know that one vertex is at (0,0) and a second one is at (0.5,0), making the length of that side 0.5. The final vertex is the coordinate where all three planes intersect.

With the equations: z = 8x/10+2y/10+d(1-x-y)/10

z = dx/10 +dy/10 +8(1-x-y)/10

z = 2x/10+8y/10+d(1-x-y)/10

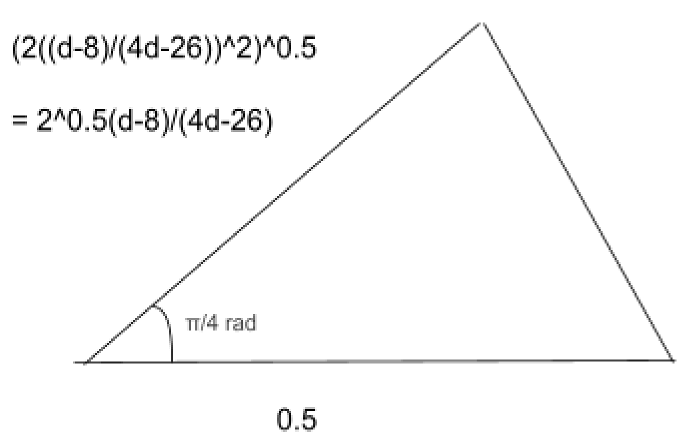
We can solve these simultaneously, and to summarise, you get coordinate

X = (d-8)/(4d-26)

Y = (d-8)/(4d-26)

Z = (d^2 -40)/(20d-130)

The change in z is unimportant, as we only care about the probability of victory depending on the change in x and y. Hence, we have a triangle with the following measurements.



Then, using the area formula A = absinC /2 (and because the quadrilateral is twice the area of the triangle), the area of the quadrilateral is (d-8)/(8d-52).
Therefore P(Green) is area/0.5 = (d-8)/(4d-26)
P(Red) or P(Blue) will equal (1-P(G))/2 which is equal to (3d-18)/(8d-52)

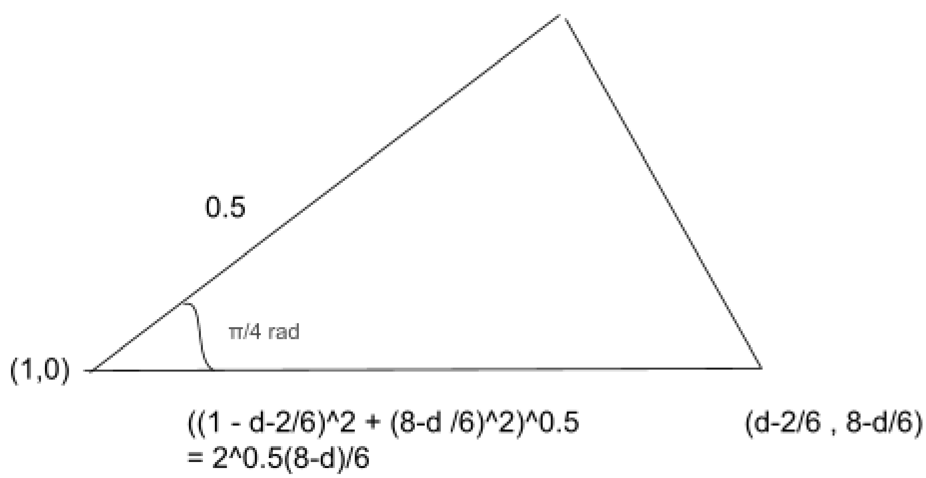
When d>5, the green area resembles a pentagon; however, the red and the blue regions resemble a triangle.

Only using the first two equations

$$z = 8x/10+2y/10+d(1-x-y)/10$$
$$z = dx/10 +dy/10 +8(1-x-y)/10$$

Using knowledge that the point of interception between one plane (red or blue) and green will fall on the line x+y = 1, we can solve the simultaneous equation and get coordinates,
X = (d-2)/6
Y = (8-d)/6
Z = d/10 (however irrelevant as we are taking top down area)

Again, we know that one coordinate will be at (1,0) and one at (0.5,0)



Again, using A = absinC /2, we see that the area of this is 8-d/2,4; hence, the probability of Red and Blue (our extremist parties) is 8-d/12. (area/0.5)

P(G) here is just 1-2(P(A)) or 1-2(P(B)). Which simply comes out as d-2/6.

We see this by plotting the probability of success of Green (Center), Blue (Extremist), and Red (Extremist) on the y-axis and the variation of d on the x-axis with bounds included.

Image from [Desmos Graphing Calculator]

This model only works for 2=< d =< 8 but as you can see, at the lowest value of d at 2/10 every party has an equal probability of victory, yet as it continues to d=5 it approaches P(G) = 0.5 and P(R) = P(B) = 0.25. It linearly increases to at d = 0.8 P(G) = 1, P(R) = P(B) = 0.

5.1 Conclusion

The rated democratic system model inherently benefits the center-most party, as its policies will each appeal to some group of people. Implementing it may have some interesting effects. For example, in the UK, since 1918, the only two parties to have a majority are the Conservatives and Labour. Using this system, the second favourite party (commonly the Liberal Democrats) may have a higher chance of gaining a majority, stopping the 2 horse race that's happened since 1918.

The conclusion you can draw from this information is up to interpretation. From one perspective, perhaps the most moderate party that satisfies people being elected is how it should be. However, one could also think that the party with the highest popular vote should be the one the people put first; it shouldn't matter who came in second, as it shouldn't influence the result.

Whatever perspective you hold, it is agreeable that our current democracy isn't perfect. Perhaps the beautiful thing is that it will never be perfect. No person is perfect, so perhaps no democratic system can be perfect either, and there will always be a sample of people who are left unfulfilled with any result.