A Tangent about Tangents

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1 Introduction

Hello dear reader, thank you for being here!

Perhaps by mentioning Pierre de Fermat, René Descartes, and (for lack of better word) a petty mathematical feud, **just perhaps** you will make the connection with the title and know what I will be writing about. And if so, you can just skip this essay entirely! But *perhaps* you don't and you're up for a little history lesson, and you'll call me a liar because "this is supposed to be an essay about mathematics", never fear, there will be plenty of maths in this essay. However, this *tangent* of an introduction is getting too long. So, let us embark on a journey through time.

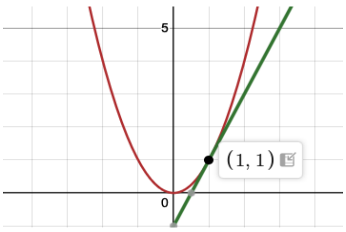
Descartes is widely renowned for his pioneering in analytical geometry, which is the ability to represent an algebraic equation onto an x-y plane, also known as the Cartesian plane (DesCARTES, CARTESian, do you see it!).

Descartes shared his findings with fellow mathematicians, which is how his work circulated into the Fermat's hands, a French Judge, who solved math puzzles in his leisure time (*gasp, one of the greatest mathematicians in history was in fact not a scholar) [1]. Anyways, Fermat criticised aspects of Descartes work, and Descartes was deeply offended that a *maths hobbyist* should criticise his work when he was so *obviously* superior, marking the beginning of a mathematical rivalry!

2 What is a Tangent

Let's take a detour to define what a tangent or tangent line is (not the function tangent of x, nor to go off on a tangent).

A tangent line, defined by Leibniz, is *"a line through a pair of infinitely close points on the curve"*. ***Well,*** that's an awfully unhelpful definition, so let's break it down. Firstly, a curve is a smoothly drawn line or shape, like a parabola or a circle. Next, what does Leibniz mean when he refers to infinitely close points? Really that is just a confusing way to say the tangent line touches the curve at one point but does not intersect it.



To visualise this, I've inserted the graph of $y = x^2$ (red line), and a straight green line which is *tangent to the curve*. We say that it is tangent to the curve since it touches $y = x^2$, at precisely (1,1) and nowhere else.

Side note: I am realising that Leibniz's definition may not be the most fitting given he was about 4 years old when Descartes died, but we're going to ignore that for now.

3 Two Differing Methods

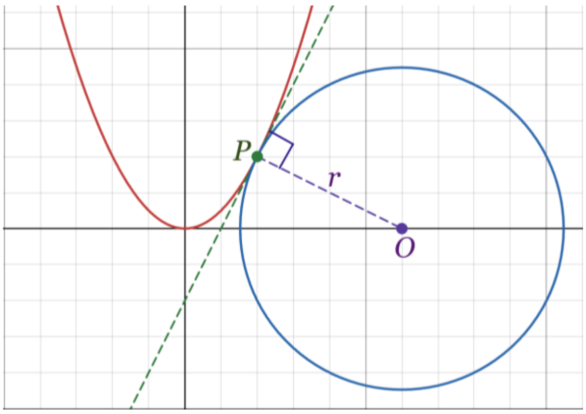
So, how do tangents come into play between Descartes and Fermat? In 1637, Descartes wrote what he considered his greatest findings in his book La Géométrie, where he penned his method to find the tangent to a curve. At the same time, Fermat had also come up with his own method of tangents. Given differential calculus wasn't invented till around 1665, both their methods are quite clever, though not to spoil the fun, but one was most definitely cleverer than the other.

I will outline each method in 3.1 and 3.2.

3.1 Descartes Method of Tangents

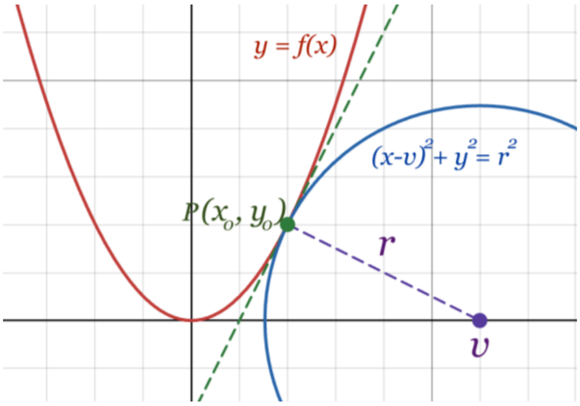
Descartes said that if we:

1. Draw a circle that touches a curve at a singular point P , with the origin (centre) of the circle on the x axis,
2. We can draw the radius of the circle from the origin to the point P . This line is normal or perpendicular to the tangent. We can see this illustrated in the figure below. Where r is the radius of the circle and O denotes the origin of the circle.



3. Now, we can solve for the x coordinate of the circle's origin. Which can help us find the gradient of the normal at point P , and thus the gradient of the tangent.

   Let's represent this mathematically. Let us define the curve to be $y = f(x)$, and the equation of a circle translated along the x axis as $(x - v)^2 + y^2 = r^2$ where v is the x coordinate of the origin, and the coordinates of point P to be (x_0, y_0).



To solve for v , we can perform a system of equations by substituting $y = f(x)$ into $(x - v)^2 + y^2 = r^2$:

$$(x - v)^2 + [f(x)]^2 = r^2$$

Rearrange and we get:

$$(x - v)^2 + [f(x)]^2 - r^2 = 0$$

Now that we have a polynomial, we can make some deductions.

a. Since the circle will touch the curve, we already know that $x = x_0$ is a root of the polynomial. (Side note: Notice how Descartes assumes that the curve will always be a polynomial)
b. Since the circle is tangent to the curve at x_0, this implies that $x = x_0$ is repeated root (if this confuses you, think about how a polynomial 'bounces off' the x axis when there is a repeated root, this is the same idea, except the curve is 'bouncing off' the tangent line).

Therefore, we can say that:

$$(x - v)^2 + [f(x)]^2 - r^2 = (x - x_0)^2\, P(x), \text{ where } P(x) \text{ is some polynomial}$$

Now, let's apply this to $f(x) = x^2$, and $x_0 = 1,\ y_0 = 1$. Let's use these values to solve for v :

$$(x - v)^2 + [x^2]^2 - r^2 = (x - 1)^2\, P(x)$$

Expand the brackets:

$$x^2 - 2vx + v^2 + x^4 - r^2 = (x^2 - 2x + 1)\, P(x)$$

If you let $P(x) = (a_2x^2 + a_1x + a_0)$, we can solve for each term of a_n based on the coefficient of x^n on the right-hand side, which I will abbreviate to RHS:

$$x^4 + x^2 - 2vx + v^2 - r^2 = (x^2 - 2x + 1)\,(a_2x^2 + a_1x + a_0)$$

$$= a_2x^4 + (-2a_2 + a_1)x^3 + (a_2 - 2a_1 + a_0)x^2 + (a_1 - 2a_0)x + a_0$$

For term x^4, RHS = 1, so $a_2 = 1$

For term x^3, RHS = 0, so $-2a_2 + a_1 = 0$

$$-2(1) + a_1 = 0, a_1 = 2$$

For term x^2, RHS = 1, so $a_2 - 2a_1 + a_0 = 1$

$$1 - 2(2) + a_0 = 1, a_0 = 4$$

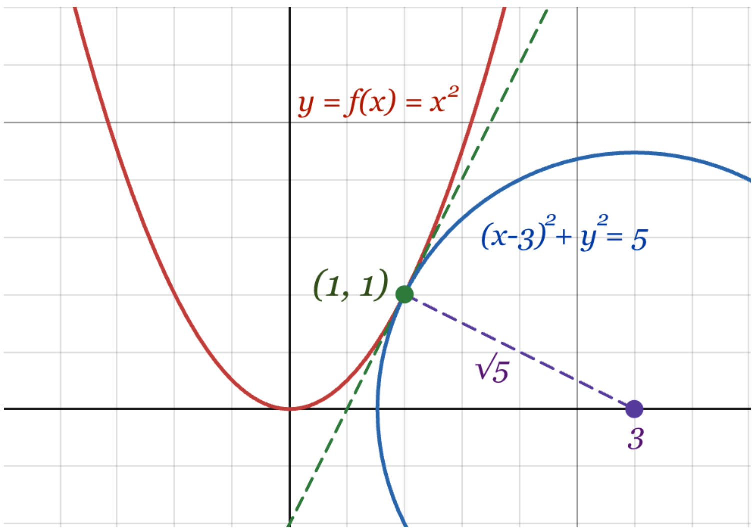
For term x^1, RHS = $-2v$, so $a_1 - 2a_0 = -2v$

$$2 - 2(4) = -2v\,, v = 3$$

For term x^0, RHS = $v^2 - r^2$, so $a_0 = v^2 - r^2$

$$4 = 3^2 - r^2, r = \sqrt{5}$$

4. Here comes the magic! Since $v = 3$, the gradient of the normal is $\frac{\Delta y}{\Delta x} = \frac{1-0}{1-3} = \frac{-1}{2}$, by taking the negative reciprocal we find that the gradient of the tangent is 2! You can see I've updated our graph below.



Through some quick substitution:

$$y = 2x + b\text{ , }b\text{ being the y intercept}$$

$$1 = 2(1) + b$$

$$b = -1$$

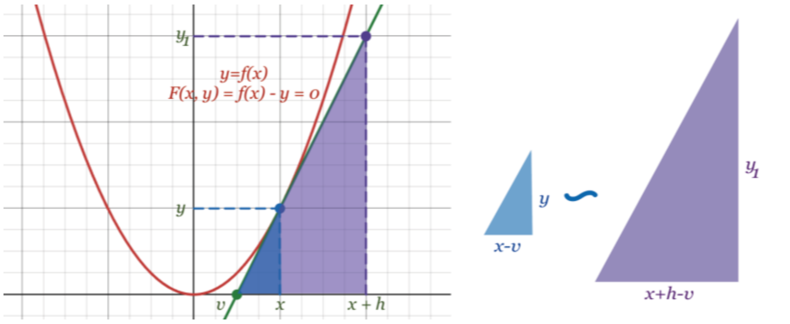
Voila! The equation of the tangent line is $y = 2x - 1$

WELL, that was long winded, imagine if we were to perform this process with higher order polynomials. It would be rather tedious, so much for *tangents*... (get it). This makes me even more grateful that Newton and Leibniz invented calculus.

3.2 Fermat’s Method of Tangents

Fermat’s method is actually quite ingenious, you’ll see as I demonstrate the steps.

1. The curve $y = f(x)$ can be rearranged and defined as $F(x,\ y) = f(x) - y\ =\ 0$.
2. We can draw a tangent line at point $(x,\ y)$, such that it intersects with the x axis at $x = v$, and reaches some point$(x + h,\ y_1)$. You can see this more easily with the figure below that Fermat has essentially created two similar triangles.



Since they are similar triangles, the ratio of the lengths of the legs are equivalent, which can help us to solve for v .

So we get the equation:

$$\frac{y}{x - v} = \frac{y_1}{x + h - v}$$

Which we can rearrange in terms of y_1:

$$y\left(1 + \frac{h}{x - v}\right) = y_1$$

3. Now Fermat says that if h gets very close to 0, at some point $(x + h,\ y_1)$will sit on the graph of $f(x)$, therefore, it can be written as $F\left(x + h,\ y_1 = y\left(1 + \frac{h}{x - v}\right)\right)$So let’s see what happens if we take apply this to the curve $y = f(x) = x^2$, such that $F(x,\ y) =\ x^2 - y\ = 0$:

$$F\left(x + h,\ y\left(1 + \frac{h}{x - v}\right)\right) = 0 = (x - h)^2 - y\left(1 + \frac{h}{x - v}\right)$$

Expand the brackets:

$$0 = x^2 - y - 2hx + h^2 + \frac{yh}{x - v}$$

Since $x^2 - y\ =\ 0$

$$0 = -2hx + h^2 + \frac{yh}{x - v}$$

4. At this point, Fermat says that since there is a common term h , we can divide the entire equation by h .

$$0 = -2x + h + \frac{y}{x - v}$$

5. Here comes the trick, since h is getting very close to zero, this means we can say $h = 0$, then solve in terms of v :

$$0 = -2x + \frac{y}{x - v}$$
$$v = x - \frac{y}{2x}$$

Side note: It is here where Fermat's method is slightly iffy, since he essentially does an illegal division by dividing the expression by h , then equating h to 0. This verges on a concept called limits that was only developed about a century later. Descartes criticised this part of his method heavily.

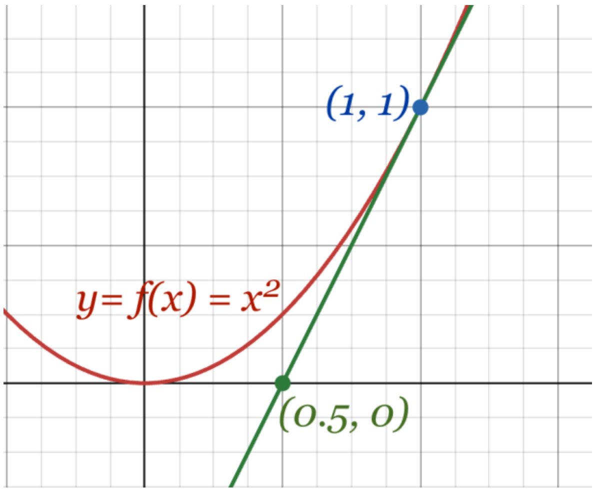
6. Let us see what happens when we substitute v into the equation for the gradient.

$$\frac{\Delta y}{\Delta x} = \frac{y - 0}{x - \left(x - \frac{y}{2x}\right)} = \frac{y}{\frac{y}{2x}} = 2x$$

Unbeknownst to Fermat, he found a way to find the derivate of a function with his method! If you're familiar with the power rule, you'll notice very quickly that $\frac{d}{dx}(x^2) = 2x$, which is the result we got using Fermat's method.

7. Now let's find the equation of the tangent at (1,1) like we did with Descartes method.

$$v = 1 - \frac{1}{2(1)} = \frac{1}{2}$$



Gradient of the Tangent Line:

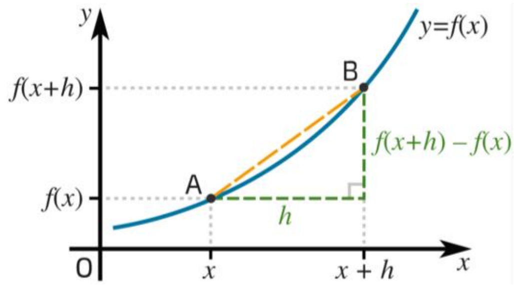
$$\frac{\Delta y}{\Delta x} = \frac{1 - 0}{1 - \frac{1}{2}} = 2$$

Through substitution you would get the equation of the tangent: $y = 2x - 1$

It's intriguing to see that the algebraic equation for v can be used to derive an equation for the gradient in terms of x . Unlike Descartes's method, Fermat's method allows us the find a general expression for the slope of the tangent at any point.

It's fascinating to see how Fermat's method so closely resembles First Principles in Differential Calculus. First Principles which I will explain simply:

If there are two points on a curve $f(x)$: $(x,\ f(x))$and $(x+h,\ f(x+h))$ we can draw a secant to join them. As h gets closer to 0, the gradient of the secant, gets closer and closer to the gradient of the tangent. We represent this mathematically as a limit:



$$\lim_{h \to 0} \quad f'(x) = \frac{\Delta x}{\Delta y} = \frac{f(x+h) - f(x)}{x + h - x}$$

$$\lim_{h \to 0} \quad f'(x) = \frac{f(x+h) - f(x)}{h}$$

We denote the slope of the tangent as $f'(x)$, which is also called the derivative.

4 Lines of Conflict

When Fermat received Descartes' method on tangents via Mersenne (another famous mathematician), he criticised Descartes for being overly complex and convoluted, and requested that Mersenne show Descartes his own simpler method.

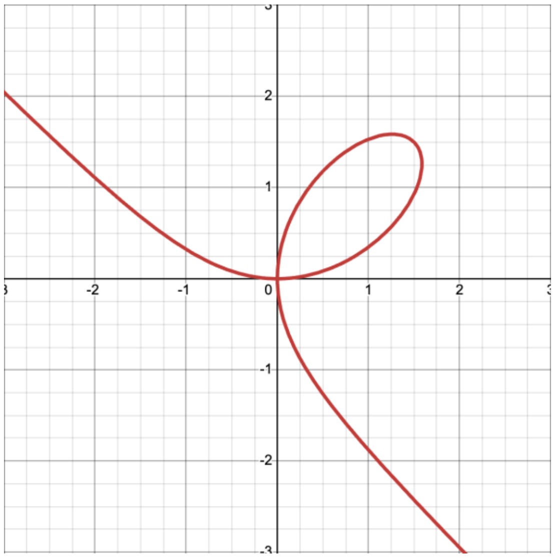
Descartes was particularly known for being a *confident* mathematician, if not a bit self-important. He abhorred the idea of Fermat attacking his work when he was more established in the field than Fermat.

Particularly, in some correspondence with Mersenne, Descartes called Fermat's method *"nothing more than a* ***false position****, founded on a means of demonstration that reduces to the impossible, and that is the* ***least valued*** *and* ***the least clever of all of that are used in mathematics****. Whereas mine is drawn from a familiarity with the nature of equations that were never explained…" [2]*

Mersenne relayed this to Fermat, who defended himself saying his method was *"just* ***as certain as the construction of the first proposition of the Elements****. Perhaps having them put forward naked and without demonstration, they were not understood, or they appeared too simple to M. Descartes, who has made so much headway and has taken such a difficult path for these tangents in his Geometry." [2]*

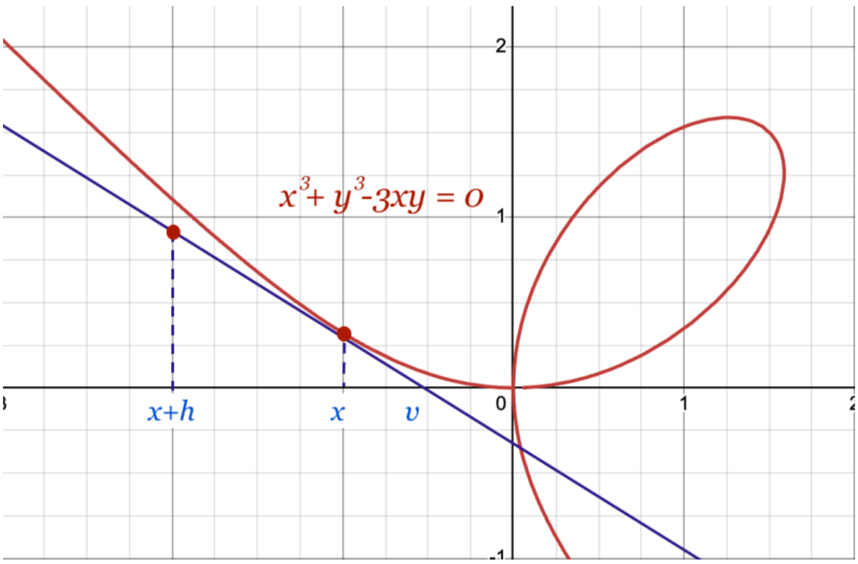
In light of such conflict, Descartes wished to disprove Fermat's method by proposing the curve, now known as the *Folium of Descartes*, and challenging him to find the tangent line on any given point as Descartes had failed to do so with his own method.

The *Folium of Descartes* is defined as: $x^3 + y^3 - 3hxy = 0$, where h is any constant, so for simplicity's sake, let's make $h = 1$.



The reason why Descartes' method breaks down is because one of the early steps in Descartes' method is to substitute $y = f(x)$ into the equation of the circle, but this is impossible with the *Folium of Descartes*, as it is defined implicitly, which means that we cannot rearrange the equation of the curve such that y is the subject.

However, Fermat was able to prove that his method still works with implicit equations. This is because his method doesn't require making y the subject, therefore allowing it to work in any type of curve.



First, we plot the point at $(x+h,\ y_1)$, which will rest on the curve $F(x,y)$ when h goes towards 0, thus: $F\left(x+h,\ y_1 = y\left(1+\frac{h}{x-v}\right)\right)$

Now we can substitute into $F(x,y) = x^3 + y^3 - 3xy = 0$:

$$F\left(x+h,\ y\left(1+\frac{h}{x-v}\right)\right) = (x+h)^3 + \left(y\left(1+\frac{h}{x-v}\right)\right)^3 - 3(x+h)\left(y\left(1+\frac{h}{x-v}\right)\right)$$

$$0 = (x+h)^3 + \left(y\left(1+\frac{h}{x-v}\right)\right)^3 - 3(x+h)\left(y\left(1+\frac{h}{x-v}\right)\right)$$

Expand:

$$0 = x^3 + 3x^2h + 3xh^2 + h^3 + y^3 + \frac{3y^3h}{x-v} + \frac{3y^3h^2}{(x-v)^2} + \frac{y^3h^3}{(x-v)^3} - 3xy - \frac{3xyh}{x-v} - 3yh - \frac{3yh^2}{x-v}$$

Since $x^3 + y^3 - 3xy = 0$:

$$0 = 3x^2h + 3xh^2 + h^3 + \frac{3y^3h}{x-v} + \frac{3y^3h^2}{(x-v)^2} + \frac{y^3h^3}{(x-v)^3} - \frac{3xyh}{x-v} - 3yh - \frac{3yh^2}{x-v}$$

Divide by common term h:

$$0 = 3x^2 + 3xh + h^2 + \frac{3y^3}{x-v} + \frac{3y^3h}{(x-v)^2} + \frac{y^3h^2}{(x-v)^3} - \frac{3xy}{x-v} - 3y - \frac{3yh}{x-v}$$

Since h goes towards 0, all terms containing h are cancelled:

$$0 = 3x^2 + \frac{3y^3}{x-v} - \frac{3xy}{x-v} - 3y$$

$$0 = x^2 + \frac{y^3 - xy}{x-v} - y$$

Solve for v:

$$v = \frac{y^3 - xy}{x^2 - y}$$

Substitute v into equation of the gradient:

$$\frac{\Delta y}{\Delta x} = \frac{y-0}{x - \dfrac{y^3 - xy}{x^2 - y}} = \frac{x - y^2}{x^2 - y}$$

Now let us verify whether $\frac{x-y^2}{x^2-y}$ is in fact equal to the derivative of $x^3 + y^3 - 3xy = 0$ with implicit differentiation. If you are only familiar with differentiating functions and not relations, let me lay this out for you:

Our goal is to differentiate with respect to x ($\frac{d}{dx}$), so we can find the derivative ($\frac{dy}{dx}$).

To do this, terms containing y will be differentiated with respect to y ($\frac{d}{dy}$) then multiplied by $\frac{dy}{dx}$, such that $\frac{d}{dy} \times \frac{dy}{dx} = \frac{d}{dx}$. Thus, we can represent our equation like this:

$$\frac{d}{dx}(x^3) + \frac{d}{dy}(y^3) \times \frac{dy}{dx} - 3\left(\frac{d}{dx}(x) \times y + \frac{d}{dy}(y) \times \frac{dy}{dx} \times x\right) = 0$$

$$3x^2 + 3y^2 \times \frac{dy}{dx} - 3\left(y + \frac{dy}{dx}x\right) = 0$$

$$x^2 - y = \frac{dy}{dx}x - \frac{dy}{dx}y^2$$

$$\frac{x^2 - y}{x - y^2} = \frac{dy}{dx}$$

Note that Fermat's found his method ineffective when trying to find the tangent line of the *Folium of Descartes* at (0, 0) since the tangent is simultaneously horizontal and vertical. Despite this, it is safe to say that Fermat's method valid, and Descartes was bested! Perhaps, you'd expect Descartes to finally accept that Fermat's method was better, but he didn't, and their feud lived on!

Descartes **continued** to speak poorly of Fermat remarking, *"They have sent me a great Register of Fermat's discoveries; but this, rather than making me think better of him or them, has reminded me that 'It's the poor man who counts his sheep'…" [3]*

After Descartes died (yes Fermat outlived him, but that's not fair to say since he was 11 years younger), Fermat lamented that he wished *"the differences that I have formerly had on [tangents] with Descartes were ended to his advantage."* Though Fermat remains polite, I almost wonder whether Fermat gets second hand embarrassment when Descartes continues to deny the validity of his method.

5 Conclusion

I hope you enjoyed this ride through mathematical history. Perhaps I strayed a bit away from tangent lines, but that's why this essay is called a tangent in the first place (sorry this joke is getting old)!

Maybe all the math we did today is completely useless, but it is fascinating to see the first inklings of differential calculus in Fermat's work and see the absolute genius of his method. It is so rare that we get to deep dive into the beginnings of mathematical ideas (outside of perhaps how we used to approximate pi) and as a lover of analytical geometry and calculus, this was very fun to research!

I find it fitting that I should write an essay about tangents, as my math teacher has made plenty of puns with my last name 'tan' and tangent lines or the tangent function. Anyway, thank you for directing your eyes normal to your computer screen to read what was on it, when we intersect (meet) again! I hope that your tangent is at the point of inflection, so we are sure to cross paths...

References

[1] Kumar, S. (2021, May 17). Fermat, Descartes, and The Dawn of Differential Calculus. Cantor's Paradise. Retrieved April 4, 2025, from https://www.cantorsparadise.com/fermat-descartes-and-the-dawn-of-differential-calculus-80472d35c0c1

[2] Skinner, L. (2015). The World Before Calculus: Historical Approaches to the Tangent The World Before Calculus: Historical Approaches to the Tangent Line Problem. WWU Honors Program Senior Projects. https://cedar.wwu.edu/cgi/viewcontent.cgi?article=1012&context=wwu_honors

[3] Descartes, R. (2013). Selected Correspondence of Descartes (J. Bennett, Trans.). Early Modern Texts. Retrieved April 4, 2025, from https://www.earlymoderntexts.com/assets/pdfs/descartes1619.pdf