Reaching the Moon with Paper

1 Introduction

We have all heard the popular thought experiment, saying that if you fold a piece of paper in half 42 times, you can theoretically reach the moon! Unfortunately, the world record is only 12 folds, where high school student Brittany Gallivan managed to acquire a 1.2km long piece of tissue. Amazingly, at 103 folds, the piece of paper would exceed the size of the observable universe by over 93 billion light years! Unsurprisingly, this thought experiment is deeply rooted in mathematics, demonstrating the mind-boggling effect of exponential growth in a tangible scenario.

In this essay, I wish to further expand on this idea of reaching the moon with paper, through a fun, mathematical lens. I will look at 2 different ways of getting to the moon using nothing but paper, and I will calculate the amount of energy we would need to carry these out (regarding manufacturing energy input, and any gravitational work done folding or stacking).

I stress that this paper should not be taken in a literal sense, as these ideas are far from feasible: I only wish to contextualise this thought experiment and see how the numbers unfold (no pun intended).

2 A paper graveyard

If you have amassed a graveyard of unused/scrap paper in your office that will never see the light of day again, why not put it to good use? My first method of reaching the moon involves simply stacking A4 pieces of paper: instead of folding the paper, the thickness of our developing structure can duplicate, by doubling the number of sheets in our stack each time we add on more paper. Admittedly, this is the more barbaric approach out of the two, but it gets the job done.

Let’s assign the thickness of one sheet of paper, x, as 0.1mm.

$$x = 0.1\ mm$$

$$0.1\ mm = 1 * 10^{-4}\ m$$

Let’s define the average distance, d, from the Earth to the Moon

$$d = 384{,}400{,}000\ m$$

We can use the simple equation, d = n * x, to find the number of sheets we need, n, to cover the average distance from Earth to the Moon.

$$3.844 * 10^{8} = n * (1 * 10^{-4})$$

$$n = \frac{3.844 * 10^8}{10^{-4}}$$

$$n = 3.844 * 10^{12}$$

We need 3.844 trillion sheets of paper... yikes. Apparently, U.S offices use 12.1 trillion pieces of paper a year. Would they notice the extra paper missing?

Now, we must look at the energy needed to manufacture this paper. In 2015, producing one ton of paper consumed (on average) 2,908 kilowatt hours of energy (A kWh is the energy consumed by a 1000-Watt appliance which operates for an hour).

We must first calculate the amount of energy required to produce one sheet of paper

The weight of one piece of paper (on average) is 0.005kg. One tonne of paper = 1000kg

$$number\ of\ sheets\ per\ ton, \qquad N = \frac{1000}{weight\ of\ one\ piece\ of\ paper\ (kg)}$$

$$N = \frac{1000}{0.005} = 200{,}000\ pieces\ of\ paper$$

To find the energy per sheet, we divide the energy per tonne by the number of sheets per tonne

$$Es\ (energy\ per\ sheet) = \frac{2{,}908\ kWh}{200{,}000} = 0.01454\ \frac{kWh}{sheet}$$

$$Total\ energy\ required = 3.844 * 10^{12} * 0.01454 = 55{,}891{,}760{,}000\ kWh$$

Using the conversion below, we can find this value in Joule

$$1\ kWh = 3.6 * 10^6\ Joules$$

$$3.6 * 10^6 * 55{,}891{,}760{,}000 = 2.01 * 10^{17}\ joules$$

For the rest of this paper, it would be useful to derive a single equation to calculate the amount of energy used up in manufacturing, as this would make the process more efficient.

$$Etotal = n * Es$$

$$where\ n = \frac{d}{x}\ (number\ of\ sheets\ needed\ to\ reach\ distance)\ and$$

$$Es = energy\ used\ up\ per\ sheet\ of\ paper$$

$$Etotal = \frac{d}{x} * \frac{2908}{N}$$

$Etotal = \frac{d*2908}{x*N}$ *where Etotal is in kWh*,

d = 384,400,000 m,

x = 0.0001 m

Which simplifies to,

$Etotal\ (Joules) = \frac{384{,}400{,}000*2908*3.6*10^6}{0.0001*N} = \frac{4.0242*10^{22}}{N}$

Now, let's assume a drone will stack these sheets of paper one on top of another. It will place one sheet at the top of the pile, fly back down to the bottom, collect another sheet, and fly back up to the top to place it. This cycle will continue until it reaches the moon. To calculate the gravitational work done, we must use the equation

$$W_n = m * g * h_n$$

Where W_n represents the work done to lift one sheet of paper, m is the mass of one sheet (0.005kg), g = 9.81 m/s^2 , and h is the height the drone travels. We only must account for the work done when the drone travels upwards, as downwards the drone is aided by gravity.

$$h_n = n * 0.0001$$

As the height displacement of the drone only increases by 0.0001m each time it stacks a sheet of paper, the relationship is linear.

To get the total work done after lifting all the sheets, we must use summation formula.

$$Wtotal = \sum_{n=1}^{3.844*10^{12}} 0.005 * 9.81 * \ n * 0.0001 = 4.905 * 10^{-6} \sum_{n=1}^{3.844*10^{12}} n$$

We can convert the sigma notation into normal algebra using the formula for the sum of the first n numbers:

$$\sum_{n=1}^{3.844*10^{12}} n = \frac{n(n+1)}{2}$$

$$Wtotal = 4.905 * 10^{-6} * \frac{(3.844 * 10^{12})(3.844 * 10^{12} + 1)}{2}$$

$$Wtotal = 3.62 * 10^{19}\ Joules$$

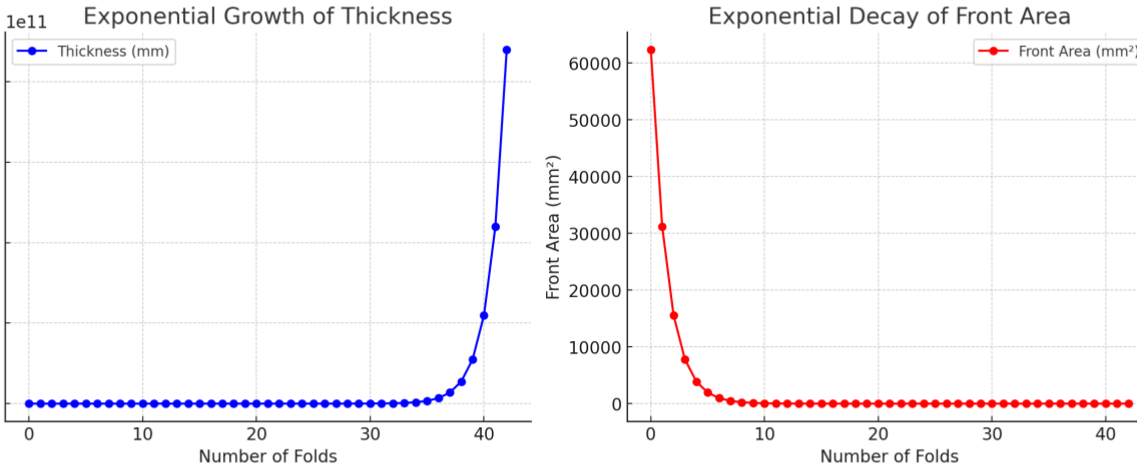
So, the total energy required to produce enough paper and stack it to reach the moon is:

$$3.62 * 10^{19} + \ 2.01 * 10^{17} = 3.64 * 10^{19}\ Joules$$

3 The ultimate piece of paper

Just like how the high school student, Brittany Gallivan, got a hold of a tissue paper 1.2 km long, we can use mathematics as our tool in trying to theoretically design a single piece of paper that can be folded enough times to reach the moon (just like in the original analogy). To simplify things, I will be using a special piece of paper with a very high folding durability, and I will assume this paper will not become too stiff to fold at any point.

Before going into calculations, it is important to understand the relationship between the thickness of the paper and the area of the front face, after each fold.



The blue diagram represents the exponential growth of an A4 paper's thickness, with the function

$$t(n) = 0.1 * 2^n$$

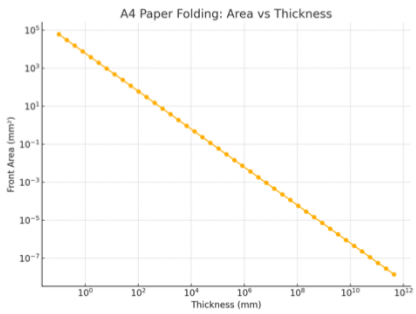
Where the initial thickness of the paper is 0.1 mm, and n is number of folds

The red diagram represents the exponential decay of the paper's frontal area, with the function

$$a(n) = 62370 * 0.5^n$$

Where the initial area of the A4 piece of paper is 62,370 mm^2.

Interestingly, if we express and plot both equations on a logarithmic scale, we can represent the inverse relationship between the two equations (same numerical value for gradient, in opposite directions) – as shown below in both graphs (the first graph is on = a logarithmic scale, the second graph demonstrated the inverse relationship between the values)

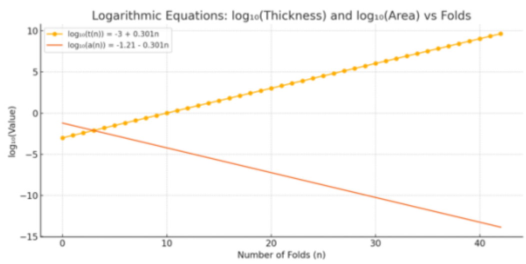


$$t(n) = 0.001 * 2^n \ (in\ meters)$$

$$Log_{10}\big(t(n)\big) = Log_{10}(0.001) + Log_{10}(2^n)$$

$$Log_{10}\big(t(n)\big) = -3 + nLog_{10}(2)$$

$$Log_{10}\big(t(n)\big) = -3 + 0.301n$$



$$a(n) = 0.06237 * 0.5^n \ (in\ m^2)$$

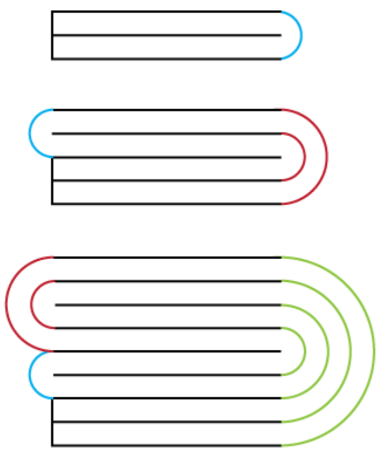
$$Log_{10}\big(a(n)\big) = Log_{10}(0.06237) + Log_{10}(0.5^n)$$

$$Log_{10}\big(a(n)\big) = -1.21 + nLog_{10}(0.5)$$

$$Log_{10}\big(a(n)\big) = -1.21 - 0.301n$$

We can derive an equation that calculates the needed ratio of the area and thickness of our starting paper. For the sake of simplicity (in later calculating the energy needed to produce this paper), I will keep the thickness of our special paper to be 0.1mm

Each fold of the paper forms a semicircle around the stacked thickness. Therefore, the minimum paper length required to make the paper fold is the circumference of the coloured semicircles below



$$circumference = \frac{\pi t}{2}$$

Where t is the thickness of the paper before the fold.

On the nth fold, the paper is being folded through 2^n layers,

So

$$total\ thickness\ right\ before\ fold\ n = 2^{n-1} * t$$

Therefore, the arc length required for fold n is

$$L_n = \frac{\pi}{2} * 2^{(n-1)} * t$$

The total length, L, of the paper needed to fold it n times is sum of all the arc length from each fold

$$L = \sum_{n=1}^{n} \frac{\pi}{2} * 2^{n-1} * t$$

$$L = \frac{\pi \mathrm{t}}{2} \sum_{n=1}^{n} 2^{n-1}$$

This is a geometric series expressed in sigma notation, so we can rewrite it as

$$L = \frac{\pi \mathrm{t}}{2}(2^{n} - 1)$$

$$\frac{L}{t} = \frac{\pi}{2}(2^{n} - 1)$$

This equation does not account for the strain of a piece of paper, or friction (sorry physicists!), and it is certainly empirical and oversimplified. Admittedly, there is a quadratic equation that accounts for these elements, modelled initially by none other than Brittany Gallivan herself.

$$L = \ \pi \mathrm{t}/6 * (2\mathrm{n}+4)(2\mathrm{n}-1)$$

But for the sake of consistency of already excluding those factors, and for simplicity, I will use the geometric equation.

Let's calculate the length of paper needed:

$$when\ n = 42, \qquad \frac{L}{t} = \frac{\pi}{2} * (2^{42} - 1)$$

$$\frac{L}{t} = 6.908 * 10^{12}$$

If t = 0.0001 m, $\qquad L = \ \mathit{690,843,530.5\ m}$

If the length of our paper is L, we can scale our width up to have the same ratio of W:L as a regular A4 piece of paper. The W:L ratio of an A4 page is about 210:297, so the width of our special paper would be

$$\frac{690{,}843{,}530.5}{297} * 210 = 488{,}476{,}223.6\ m$$

When calculating the energy required for this method of reaching the moon, we must factor the work done folding the paper (from the weight of the paper only, not the stiffness or friction), and the manufacturing of the paper.

We can use the equation we created above in this situation, to calculate the energy required to manufacture the paper:

$$Ejoules = \frac{4.0242 * 10^{22}}{N}$$

Where N = the number of standard A4 sheets of paper in one metric ton.

We will calculate N by seeing the equivalent number of standard A4 papers of area 0.06237 m^2 inside our big paper.

Our big paper has an area of

$$690{,}843{,}530.5 * 488{,}476{,}223.6 = 3.375 * 10^{17}\ m^2$$

(almost 2.3% of the Earth's land)

There are

$$\frac{3.375 * 10^{17}}{0.06237} = 5.41 * 10^{18}$$

$5.41 * 10^{18}$ *sheets of normal paper within our big paper.*

$$N = \frac{1000}{5.41 * 10^{18}} = 1.848 * 10^{-16}$$

Therefore,

$$Ejoules = \frac{4.0242 * 10^{22}}{1.848 * 10^{-16}} = 2.177 * 10^{36}\ J$$

And we haven't even looked at the work done yet!

The main theme of exponential trends carries on through the force needed to fold the paper in more ways than one: The number of layers being lifted per fold increases exponentially, and therefore, the vertical displacement of each fold increases exponentially, as each fold stacks on top of the previous ones.

To calculate work done folding, we must find the total mass of our paper to begin with. We can use the equation below:

$$\rho = \frac{m}{V}$$

Where,

$$V = t * W * L = 0.0001 * 488{,}476{,}223.6 * 690{,}843{,}530.5$$

$$V = 3.375 * 10^{13} m^3$$

$$\rho = 800 \frac{kg}{m^3} \ (density\ of\ typical\ office\ A4\ paper)$$

$$m = \rho V$$

$$800 * 3.375 * 10^{13} = 2.7 * 10^{16} kg$$

For each fold, the work done can be calculated with

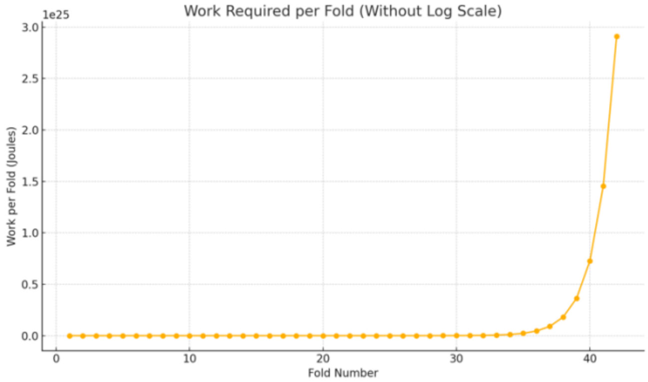
$$W_n = m_n * g * h_n$$

$$where\ h_n = t * (2^n - 1) - the\ vertical\ displacement\ of\ the\ paper, (due\ to\ the\ fold)$$

$$m_n = 0.5^n * m * 2^{n-1} = \frac{m}{2}$$

$\frac{m}{2}$ is the mass being lifted each fold. (As the area being lifted halves each time, the thickness doubles. Therefore, the exponents cancel, and mass being lifted is constant!).

$$W_n = \frac{m}{2} * 9.81 * 0.0001 * 2^{n-1} = m * 4.905 * 10^{-4} * 2^{n-1}$$



The work done for each fold also has an exponential trend, as demonstrated on the graph.

To find the total work done across all the folds, we must use the summation formula again

$$Wtotal = \sum_{n=1}^{42} m * 4.905 * 10^{-4} * 2^{n-1}$$

$$= m * 4.905 * 10^{-4} \sum_{n=1}^{42} 2^{n-1}$$

$$\sum_{n=1}^{42} 2^{n-1} = \frac{2^{42} - 1}{2 - 1} = 2^{42} - 1$$

Therefore,

$$Wtotal = m * 4.905 * 10^{-4} * (2^{42} - 1)$$

$$W_{total} = \frac{2.7 * 10^{16}}{2} * 4.905 * 10^{-4} * (2^{42} - 1) = 5.81 * 10^{24}\ Joules$$

So, the total energy required to produce this paper and fold it in half enough times to reach the moon is:

$$5.81 * 10^{24} + 2.177 * 10^{36} = 2.177 * 10^{36}\ Joules$$

4 A direct comparison

Clearly, the folding method - as fun as it is - is a lot less energy efficient than the stacking method ($2.177 * 10^{36}\ Joules$ vs $3.64 * 10^{19}\ joules$ respectively). We can represent this disparage on a logarithmic scale, with the equation below:

$$Log_{10}(Efold) - Log_{10}(Estack) = Difference\ in\ order\ of\ magnitude$$

$$Log_{10}(2.177 * 10^{36}) - Log_{10}(3.64 * 10^{19}) = Log_{10}(5.98 * 10^{16}) = 16.78$$

We can also explore the relationship (for both methods) between useful energy (the energy directly going towards our goal of lifting/folding paper to reach the moon) and total energy input.

$$\%\ Energy\ Efficiency = \frac{Useful\ Work\ Output}{Total\ Energy\ Input} * 100$$

$$stacking\ efficiency = \frac{3.62 \times 10^{19}}{3.6401 \times 10^{19}} * 100 = 99.45\%$$

$$folding\ efficiency = \frac{5.81 * 10^{24}}{2.177 * 10^{36}} * 100 = 2.67 * 10^{-10}\%$$

Not only is the stacking method approximately $10^{16.78}$ times more energy efficient, but it's ratio of useful: total energy is $3.72 * 10^{11}$ times ($\frac{99.45}{2.667*10^{-10}}$) greater!

5 Conclusion

Although this began with me merely entertaining imagination as I stretched, pushed, and prodded this silly thought experiment to the mathematical limit, we clearly saw the unexpected disproportion that has arisen towards the end, which inevitably leads to broader questions on the nature of mathematics.

This importantly achieves my goal of using a tangible situation to demonstrate the staggering disparity between linear and exponential growth. Both methods eventually reach the same distance, but at what cost? One must travel millimetre by millimetre, whilst the other makes 3 significant leaps to the finish line. We have undeniably explored the more unintuitive side of maths, enabled by my daydreaming.

The stacking method, whilst clearly excruciatingly slow and inefficient, embodies linear growth; Its strength lies in its inherent weakness, as the monotonous act of stacking the paper results in steady, predictable growth. The folding method, though elegant and creative in theory, rapidly spirals out of control in energy requirement, due to the monstrous exponential growth interwoven in most of its aspects: the area, thickness, and work done, to name a few.

By contextualizing both methods, this paper aimed not only to entertain, but to demonstrate what mathematics does best: It lets us dive into and probe the depths of any idea, and it never fails to surprise.

Let's just hope that there's no wind blowing – even the best math falls apart when reality hits.