The sizeless set meets mathematical art

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1 Introduction

Fractals are a form of algorithmic, mathematical art that involves a repetitive pattern when observed at an increasingly small scale [1]. Although it had been explored by mathematicians long before, the concept of fractals gained prominence in the 1970s when Benoit Mandelbrot popularized the idea of self-similar, complex patterns found in nature and mathematics. Notable fractals include the branching patterns of trees and ferns, the spiral arrangement of seashells, the Mandelbrot set, and the Koch snowflake [2].

This essay initially explores a very notable and fascinating set in pure mathematics, the Cantor set, which oddly combines being sizeless and infinitely uncountable. This subsequently motivates the idea of a fractal, allowing us to see how partitioning sets is a method of generating mathematical art by viewing the Cantor set as a fractal and its generalization, the Cantor dust. Finally, we will briefly examine how the Cantor set can help us identify the chaotic nature in linear maps, which is inextricably linked with fractals.

2 Countability of infinite sets

In order to better grasp how unique and special the Cantor set is, we have to review the definition of countable infinite sets:

Definition: An infinite set A is called countable if its elements can be put into an ordered list: $A = \{a_1, a_2, a_3, ...\}$. More formally, the infinite set is countable if there exists a bijection $f : A \to \mathbb{N}$.

From the definition, it follows immediately that the set of natural numbers, $\mathbb{N}$ is countably infinite. We can then prove that the integers and the rationals are also countable (by appropriately defining a bijection, f) and that the reals are uncountable (by Cantor's diagonal argument). The proofs are not provided here as they exceed the focus of this essay but they have been included in the appendix.

3 The Cantor Set

Let's now use these concepts and our intuition to examine the very unique Cantor set. It is constructed by defining a sequence of nested subsets of the real line. We initially consider the interval: $\Delta_0 = [0, 1]$.

Then, we remove the middle third of this interval giving the set: $\Delta_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. The middle thirds of these two intervals are then removed yielding: $\Delta_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. We continue this process of removing middle thirds indefinitely to obtain a sequence of sets: $\Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \Delta_3$ ...

We can then define the Cantor set as: $\Delta = \bigcap\limits_{k=1}^{\infty} \Delta_k$. That is the set of points $x \in \Delta_k$ for all k. In other words, the set of points that remain after the process of removing middle thirds has been repeated infinitely many times.

Is there anything left after this process? We can easily see that the total length of sections removed is: $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \ ... = \frac{1}{3} \sum\limits_{n=0}^{\infty} (\frac{2}{3})^n = 1$, using an infinite geometric series. Therefore, the total length removed is 1, which is equal to the size of the entire interval $[0, 1]$. Hence, it follows that the Cantor set has zero size!

This appears very counterintuitive because, evidently, the Cantor Set has elements, which we can present very elegantly for that matter. Clearly, the points that remain are the end-points of each interval Δ_k because the process of removing middle thirds leaves them intact. For instance, the points $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}$ are in the Cantor set.

We can significantly simplify the removal process by using the ternary representation. Numbers x in the interval $[0, 1]$ are represented in ternary in the form: $\sum\limits_{n=1}^{\infty} \frac{a_n}{3^n}$, where $a_n \in \{0, 1, 2\}$. The point is that this representation is a successive partitioning into a sequence of intervals each a third the size of the previous one. The values of a_n indicate whether the number lies in the first, middle, or third section of the interval. This implies that by removing the middle third, we are essentially not allowing $a_n = 1$. Hence, the elements of the Cantor set are all numbers whose ternary representation only contains 0s and 2s.

The fact that the Cantor set contains elements yet has zero size already seems contradictory by itself. Now let's make it even more paradoxical (yet more elegant than ever) by using the notion of (un)countability with the Cantor set. From the aforementioned conclusion, we understand that it involves the strings of just two numbers, which is clearly equivalent to the binary representation, which is a representation of $\mathbb{R}$ that we know is uncountable (see appendix).

Therefore, we have surprisingly concluded that the Cantor set is both uncountable and has zero size.

4 Fractals

Now that we have got a grasp of the Cantor set, the next step is to explore fractals. The truth is that it is actually not easy to provide a strict definition of what a fractal is. Generally, we can say that a fractal is an irregular geometric object with an infinite nesting of structure at all scales, or (even more generally...) a geometric shape containing detailed structure at arbitrarily small scale [1].

While these definitions give an intuitive understanding of what constitutes a fractal, we can still make this more rigorous.

4.1 Hausdorff Dimension

Naturally, we use the notion of "dimension" either in Euclidean space or in the variables of a dynamic system.

Since fractals are irregular geometric objects, our intuition of dimension will not work. Let us take an object residing in Euclidean dimension D and reduce its linear size by $\frac{1}{r}$ in each spatial direction. Its measure (length, area, or volume) would increase to $N = r^D$ times the original. The figure below illustrates some examples of how this applies to basic Euclidean shapes/solids [3]:

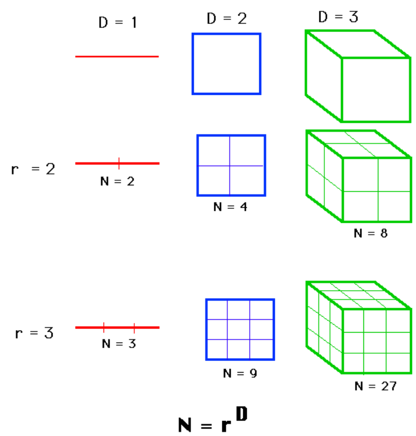

Figure 1: The concept of dimension applied to basic Euclidean shapes/solids. [3]

Now let us solve this formula in terms of D: $\log N = D \log r \Leftrightarrow D = \frac{\log N}{\log r}$. Evidently, D does not have to be an integer here, as in Euclidean geometry. This is the case in fractal geometry.

4.2 Properties of Fractals

We have already presented some of the properties of these beautiful geometrical objects. Now it is time to examine them more systematically. Properties of fractals include [4]:

- Self similarity, which can incorporate:
  - **Exact self-similarity**: a fractal is identical from all scales, such as the Koch Snowflake.

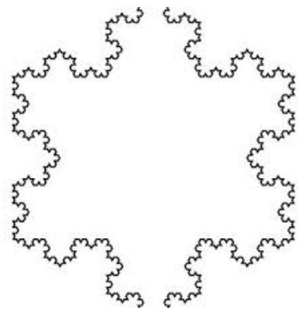

Figure 2: The Koch snowflake fractal. [5]

- **Quasi self-similarity**: a fractal approximates the same pattern at different scales; it may contain small copies of the entire fractal in distorted and degenerate forms. For instance, the Mandelbrot set's satellites are approximations of the entire set, but not precise copies.

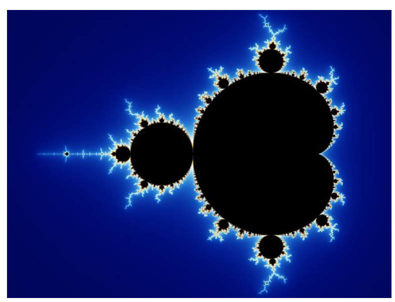

Figure 3: The Mandelbrot set fractal. [6]

- **Statistical self-similarity**: a fractal repeats a pattern stochastically so numerical or statistical measures are preserved across scales. Randomly generated fractals like the coastline of Britain (a section of the appendix is devoted to this interesting paradox), for which one would not expect to find a segment scaled and repeated as neatly as the repeated unit that defines fractals like the Koch snowflake.
- **Qualitative self-similarity**: as in a time series.
- **Multifractal scaling**: a fractal is characterized by more than one fractal dimension or scaling rule.

• Fine or detailed structure at arbitrarily small scales: A consequence of this is that fractals may have emergent properties.

• Irregularity locally and globally that cannot easily be described using "traditional" Euclidean geometry other than as the limit of a recursively defined sequence of stages.

4.3 Cantor Set as a Fractal

Let us now explore how we can unleash the beauty of the Cantor set by studying its fractal nature. We will begin by determining its Hausdorff Dimension. We have: $D = \frac{\log 2}{\log 3} \approx 0.6309 \notin \mathbb{Z}$ and therefore we can already deduce that the Cantor set manifests fractal properties!

Now it is finally time to visualize the Cantor set as a fractal. Start by representing our initial interval ($\Delta_0 \equiv [0, 1]$). as a segment of length 1. We will then visually follow the same process of parsing the initial and the sub-intervals while omitting the middle thirds each time, as illustrated in the figure below. Infinitely continuing this process yields the Cantor set fractal, which has the property of exhibiting exact self-similarity.

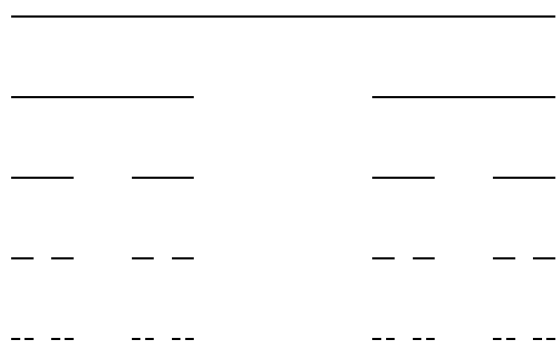

Figure 4: The Cantor set fractal created by partitioning the $[0, 1]$ interval (created using Python).

We have just shown how a sizeless and yet uncountably infinite set can produce a fractal!

4.4 Cantor dust

Let us make our fractal more aesthetically pleasing by adding dimensions (referring to our "traditional" Euclidean sense of dimension). We can hence generalize the Cantor set fractal into the Cantor dust. To generate this, we commence with a square and follow the same iterative process as before by removing the middle third horizontal and vertical stripes of each square [7], as illustrated below:

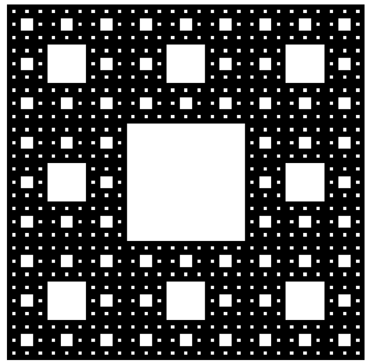

Figure 5: The Cantor dust fractal created by the iterative sequence explained above (created using Python).

The Hausdorff dimension of Cantor dust is: $d = \frac{\log 4}{\log 3} \approx 1.261859$, again showing that Cantor dust is a fractal with exact self-similarity.

Interestingly enough (we do not quite have the space to cover this completely in this essay) a fractal analysis method based on Cantor dust can be used to describe the regularity of fracture patterns in geological materials [8].

5 Cantor set fractal and Chaos

The purpose of this final section is to illustrate how the Cantor set fractal can be used to identify chaotic nature in the dynamics produced by piecewise linear maps. This reinforces the intrinsic and beautiful link between fractals and chaos in dynamical systems.

5.1 Chaos

A map alongside an initial seed value can produce a dynamics through an iteration scheme. For instance, if we consider an iteration scheme like: $x_{n+1} = f(x_n)$ and put an initial seed value of x (say x_0), the map can generate a dynamics where the trajectory will be formed by discrete points, which will be produced by the repeated iteration of the map.

In general, the map $f(x)$ should at least be quadratic in order to have chaotic behaviour in dynamics produced this way. However, piecewise linear maps can generate qualitatively different kinds of behaviours and even culminate in chaotic or unpredictable situations in a bounded domain.

The Cantor set can play a vital role in studying this unpredictable behaviour, as fractals constitute the signature of chaos. Specifically, when we observe self-similarity in the dynamics of a system, we can suspect deterministic chaos [9].

5.2 Cantor-like sets with different partitioning

If you found the decision to partition $[0, 1]$ and the subsequent intervals into thirds arbitrary this section is for you. Now starting from $[0, 1]$, we will see how we can partition our initial interval differently creating Cantor-like sets with fractal properties.

For $k \in (0, 1)$, we will use the notation $C(k)$ to denote the set that is constructed through the same iterative process as the Cantor set, but now removing the middle k^{th} interval (for the Cantor set $k = \frac{1}{3}$ and we are removing the middle third interval of our partition which has size $\frac{1}{3}$). In this section we will illustrate that for all $n \in \mathbb{N}, n \geq 3, n \neq 4$, $C(\frac{n-2}{n})$ behaves like a fractal.

To comprehend the self-similarity and the infinite detail of $C(\frac{n-2}{n})$ all we have to do is simply visualize the outcome of the process of partitioning, which is very similar to the Cantor set fractal. Below we have indicatively included $n = 5$:

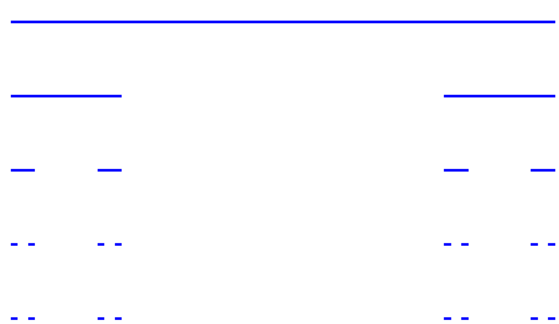

Figure 6: The fractal that is created by following the same iterative process as with the Cantor set, but with fifths (created using Python).

For the most important test, let us calculate the Hausdorff dimension of $C(\frac{n-2}{n})$. We have: $d = \frac{\log 2}{\log(\frac{n}{2})}$. We know that the logarithm is an increasing function and therefore the Hausdorff dimension will necessarily be non-integer when: $\frac{n}{2} > 2 \Leftrightarrow n > 4$ (for $n = 3$ we can easily see via substitution that we have a fractal dimension). In terms of the "quality" and intricacy of our fractal, this will increase with the complexity of the Hausdorff dimension (e.g. when $n = 5$, $d \approx 0.7564708$, whereas when $n = 8$, $d = 0.50$ and hence the first fractal will be more complex).

This helps us see how partitioning sets is in general a technique for generating fractals beyond the Cantor set.

5.3 Example of identifying chaos in a linear map

Let us take the following example: consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 2\lambda x & \text{if } x \leq \frac{1}{2} \\ 2\lambda(x-1)+1 & \text{if } x > \frac{1}{2} \end{cases}$$

where $\lambda > 0$ is a control parameter [9].

From the definition of f, it is clear that: $f^n(x) \to \pm\infty$ as $n \to \infty$ for $x \in (-\infty, 0) \cup (1, +\infty)$, where $f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x)$, so no interest is left out in this case as the trajectories formed by the dynamics of the map escape out to $\pm\infty$.

However, if we were to consider the dynamics produced by f for points within the interval $\Delta_0 \equiv [0,1]$, we will have a completely different qualitative behaviour. The points in Δ_0 which are mapped by f onto either of the intervals $(-\infty, 0)$ and $(1, +\infty)$ will diverge to $\pm\infty$ upon subsequent itertions of the map.

After the first iteration, the points in $F_1 = [0, \frac{1}{2\lambda}] \cup [1 - \frac{1}{2\lambda}, 1]$ are mapped to [0,1], whereas points in the middle open interval $F_1^{'} = \left(\frac{1}{2\lambda}, 1 - \frac{1}{2\lambda}\right)$.

Upon the second iteration, the points on the set $F_2^{'} = (\frac{1}{4\lambda^2}, \frac{2\lambda-1}{4\lambda^2}) \cup (\frac{4\lambda^2-2\lambda+1}{4\lambda^2}, 1 - \frac{1}{4\lambda^2})$ are mapped to $F_1^{'}$ and hence they will diverge to $\pm\infty$. The points on the set $F_2 = [0, \frac{1}{4\lambda^2}] \cup [\frac{2\lambda-1}{4\lambda^2}, \frac{1}{2\lambda}] \cup [1 - \frac{1}{2\lambda}, \frac{4\lambda^2-2\lambda+1}{4\lambda^2}] \cup [1 - \frac{1}{4\lambda^2}, 1]$ are mapped to the set F_1.

We then continue these iterations indefinitely.

We can conclude that the attractor set of f is simply the set $C(\frac{n-2}{n})$ where $n = 2\lambda$. However, we have already shown that $C(\frac{n-2}{n})$ is a fractal and hence we can conclude that the map f manifests chaotic behavior.

We can observe the chaotic behaviour by plotting a bifurcation diagram and a cobweb diagram.

"Bifurcation" implies a qualitative change in the dynamics that occur in a system with the variation in the parameter. A "bifurcation diagram" is a diagram where the orbit points generated by the dynamics converge to, say, x are plotted against the parameter values. The bifurcation diagram helps us to visualize different types of dynamics that take place in a system with the variation in the parameter.

The bifurcation diagram for our linear map, f is shown below:

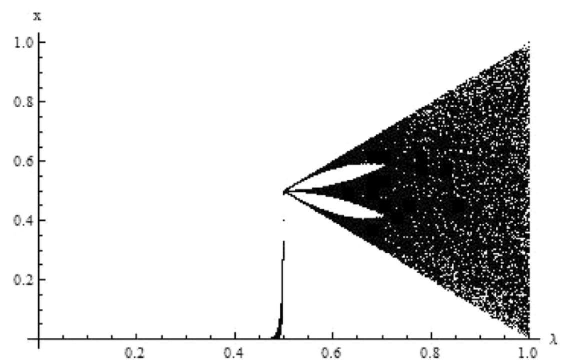

Figure 7: Bifurcation diagram for out map, f. We can see how convergence depends on the value of control parameter, λ. [9]

Note that the bifurcation diagram terminates after $\lambda = 1$, where a boundary crisis occurs. That is, after $\lambda = 1$, there exists an escape region in which if a trajectory lands, the subsequent iteration then rapidly takes the trajectory off toward $-\infty$. The bifurcation diagram shows that upon approximately $\lambda = 0.7$ chaos creeps into the dynamics produced by f which is marked by areas having Cantor-like dust.

We can further verify the chaotic nature of the map with the aid of a cobweb diagram. A cobweb diagram is a geometrical representation of the iteration scheme by which the dynamics is produced. The cobweb diagram that we provide corresponds to a control parameter $\lambda = 0.8$ by taking 300 iterations with the initial seed value $x_0 = 0.3$, which also confirms that we have a chaotic situation for this value of λ:

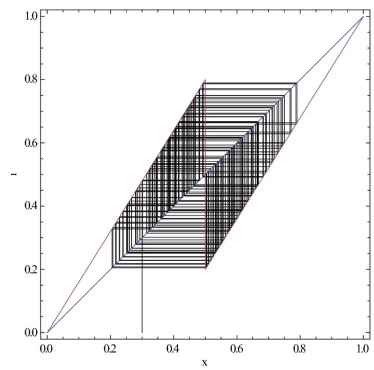

Figure 8: The cobweb diagram for our linear map with control parameter $\lambda = 0.8$, generated upon 300 iterations. [9]

6 Conclusions

The Cantor set is a truly beautiful mathematical construction, which elegantly combines being sizeless and uncountably infinite. While this set is spectacualar in its own right, what makes it even more interesting is the manifestation of fractal properties by its multidimensional equivalent, the Cantor dust. The outcome of this partition motivates more techniques to generate fractals. Finally, the application of the Cantor set and its generalization for the identification of the chaotic nature in linear maps further highlights how powerful it is. It is truly fascinating that such a simple partition of the elementary unit interval can yield so many results.

References

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7 Appendix

The objective of this essay is to incorporate some of the elegant proofs that were not included in the main body of this essay and some additional information that diverges from the focus of this essay.

7.1 Countability of $\mathbb{Q}$

We will proceed by creating a bijection $f : \mathbb{Q} \to \mathbb{N}$. Our definition of the rational numbers is all $x = \frac{m}{n}$, where $m, n \in \mathbb{Z}$. To prove countability, it just suffices to focus on the positive rationals. Therefore, we are essentially just looking for a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. To attain this, we will be using the aid of number theory, and specifically the prime factorization.

We can define the function $f(n, m) = 2^n 3^m$, which is obviously injective as prime factorizations are unique and surjective onto a subset of the naturals (we can modify our arrangement so that we cover all the natural numbers). It therefore follows that the set of rational numbers is indeed countable.

7.2 Uncountability of $\mathbb{R}$

We will now be proving that the $\mathbb{R}$ is uncountable. To attain this, we will be using a simple yet very elegant approach called Cantor's diagonal argument.

As explained in section 1, one of the primary properties of countability is that the elements of the set can be written as an ordered list. For the sake of contradiction, assume that the real numbers, r_i, can be put into an ordered list:

$$r_1 = m_1.\textcircled{a_{11}}a_{12}a_{13}\dots$$

$$r_2 = m_2.a_{21}\textcircled{a_{22}}a_{23}\dots$$

$$r_3 = m_3.a_{31}a_{32}\textcircled{a_{33}}\dots$$

and so on, where $m_i \in \mathbb{N}$ and $a_{ij} \in \{0, 1, 2, \ ...\ 9\}$. We will now construct the number $x = 0.b_1b_2b_3...$, where $b_i \neq a_{ii}$ (and hence the diagonal nature as can be seen in the list above). We can therefore conclude that the number we have constructed differs from r_n in the n^{th} digit. Thus, it is not contained in the list and hence we have constructed a real number that is not in our list, a contradiction.

Hence, the set of real numbers is uncountable.

7.3 Coastline of Britain fractal

Fractals are elegant as is, but what makes them even more beautiful is their emergence in nature. A prominent example is the coastline of Britain fractal.

In reality, however, this is a paradox. As seen in the image below, a landmass (here, Britain) does not have a well-defined length. This is attributed to the fact that coastlines manifest fractal curve-like properties. It is confirmed by the fact that the coastline has fractal dimension (aka Hausdorff dimension).

Figure 9: Two different ways of measuring the length of Britain's coastline. [10]

It turns out that the coastline of Britain (and all coastlines for that matter) belong in a group of fractals that have a Hausdorff dimension between 1 and 2 (and typically less than 1.5). Mandelbrot suggested that the coastline can be modeled by: $L(\epsilon) \approx F\epsilon^{1-D}$, where L is the length of the coastline, a function of the measurement unit, ϵ (in a sense the resolution of the length measurement), the Hausdorff dimension, D, and constant F. We can rearrange this expression to get: $L(\epsilon) = F\epsilon^{D-1}\ \epsilon$, which implies that the length consists of Fe^{-D} "units" of ϵ.

To get a better grasp of the coastline as a fractal, here are some typical values of the Hausdorff dimension of different coastlines: South Africa: $D = 1.02$, west coast of Great Britain: $D = 1.25$, and for lake shorelines, typically: $D = 1.28$ [10].