The Not-So-Solitary Mathematician: Erdős, Graphs, and How Connected We Truly Are!

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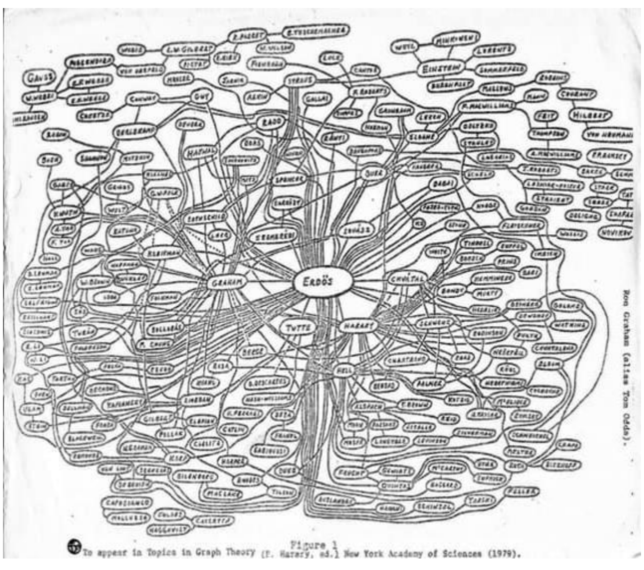
The mathematician; socially awkward, introverted, and solitary. Almost always portrayed in this way in media, it's hard to see how mathematics can be one of the most collaborative fields in academia. Thankfully, while we mathematicians worked on the Big Bang Theory, we aren't all characters in it (sorry Sheldon!), so the stereotypes don't ring universally true. Known to appear with nothing but a bag on the doorsteps of his colleagues, with no home of his own, Paul Erdős made a name for himself as the most academically social mathematician of all time – despite reportedly being a terrible houseguest!

The Erdős Number

Famously the most published mathematician of all time, Paul Erdős wrote around 1500 mathematical articles in his lifetime. He is only rivalled by Euler himself, who published fewer papers, but more pages of his work. He worked with 509 collaborators on these, and those collaborators worked with many more people in their lifetimes, and so on. First started by some of Erdős's friends, as a joking reference to his prominence, mathematicians began calculating their 'Erdős Number' – namely, how many co-authors away they were from Erdős himself.

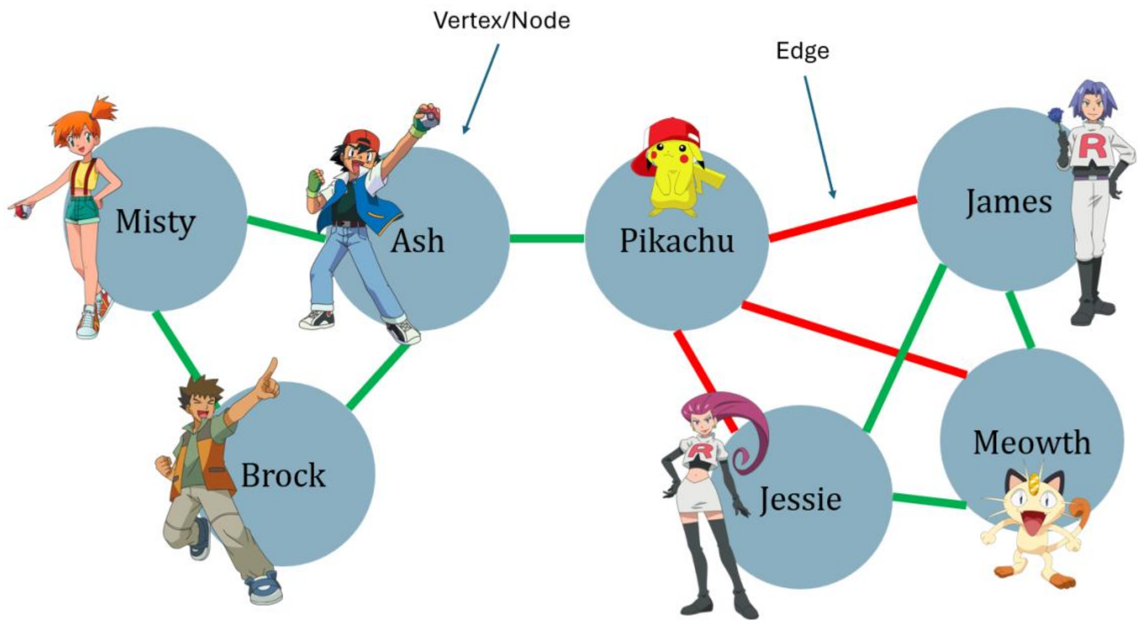
So, if I had published with Erdős himself, my number would be 1. Anyone to publish with me would then have a number of 2, and anyone to publish with them would have a number of 3 and anyone to publish with them would have a number of 4 and- I'll be here forever! Or will I? As it turns out, excepting authors with an Erdős Number of infinity (where there is no path back to him), no-one has an Erdős number higher than 15. In fact, only 2% have a number higher than 8, and the median comes out at 5 – remarkably low for a network of 268,000 people. And so, the solitary creature that is the mathematician suddenly seems far more social – go tell your friends!



Graphs: Models of Relationships

So how does this come to be? Of 268,000 mathematicians with links to Erdős, how is it that no-one is more than 15 links away? This takes us onto graphs – a topic Erdős himself contributed heavily to.

So, what is a 'graph'? You may think that's a silly question – you'll have met graphs in primary school! But here, we don't mean a *general* graph, like a scatter graph or bar chart. A graph is essentially a network of links between items. They're used everywhere, from social media to public transport to neural networks. To explain the basic nature of a graph, I'll use an example from the Pokémon anime series, to honour the number of ways graphs can be used in the franchise (if interested, I highly recommend looking into it!).



Let each vertex represent a distinct character, and each edge represent a relationship between the characters it connects. Here, we have green edges to represent a friendship, and red edges to represent hostility.

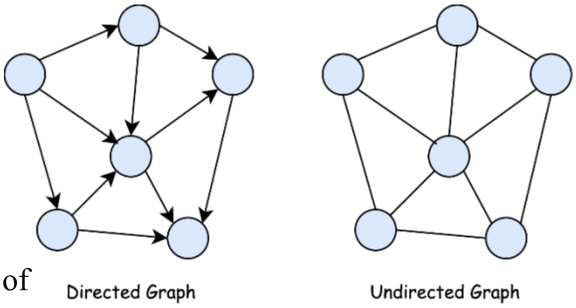
A graph like this makes it easy to see that we have two clear friendship groups (Misty/Ash/Brock & Jessie/James/Meowth), as well as Ash's companionship with Pikachu, who's enemies with the Team Rocket group. Using vertices and edges makes this far easier to understand than a verbal explanation as the graph gets larger, as well as allowing you to scan certain vertices' edges more easily.

Graphs can be defined using the shorthand

$$G(V, E)$$

where V represents the set of vertices (nodes) and E represents the set of edges (connections between nodes).

There are two key kinds of graphs we need to know about: **undirected and directed**. The example we've met is an *undirected graph* – edges between vertices have no direction, and the connection goes both ways i.e. James is friends with Jessie and Jessie is friends with James. In a *directed graph*, edges can be thought of as an arrow instead, with a distinct direction between nodes i.e. Ash catches a Pokémon, but the Pokémon wouldn't catch Ash. Graphs of both types are seen all around us - consider social media. On LinkedIn, the popular networking site, when you form a connection with someone, they also form a connection with you – it's two-way! For this reason, LinkedIn can be depicted as an *undirected graph*. Similarly, Instagram can be depicted as a *directed graph*. If I follow Billie Eilish, that doesn't mean she follows me (as much as I wish!). We'll be focussing on undirected graphs, but the properties of directed graphs are very similar.



Using our knowledge of undirected graphs, for the example above, we can define G(V, E) as:

V = {Misty, Ash, Brock, Pikachu, Jessie, James, Meowth}

E = { {Misty, Ash},
{Misty, Brock},
{Ash, Brock},
{Ash, Pikachu},
{Pikachu, Jessie},
{Pikachu, James},
{Pikachu, Meowth},
{Jessie, James},
{Jessie, Meowth},
{James, Meowth} }

Each edge is a set of the two vertices it connects. E is therefore a set of sets of vertices.

In undirected graphs like this, the sets of vertices are unordered i.e. {a, b} = {b, a}.

In directed graphs, the sets would be in the order that the arrow points.

This example has 7 vertices and 10 edges in total.

We can also describe the number of neighbours a node has as its **degree**. In our example, Misty has two neighbours (Ash and Brock), so has a *degree* of 2. This will be useful later!

We can also describe the length of the shortest path between two nodes as its **distance (d)**. We can see that the shortest path between Misty and Jessie is Misty to Ash to Pikachu to Jessie. This gives Misty and Jessie a *distance* of 3. Each graph also has a **diameter**, which is the largest distance between any of its two nodes – for our example, this is also 3, as the path between Misty and Jessie is the longest.

Now we know what a graph is, we can start to understand how Erdős Numbers come about. Each author can be represented as a vertex, connected to each of their co-authors by an edge, and voilà – we have a graph. Erdős Numbers are simply the *distance* between Erdős and the mathematician!

So, we know how the greatly coveted (amongst mathematicians anyway) Erdős Number comes about, but why should you care? Odds are, you aren't a professional, published mathematician, so how does this relate to you? Well, what if I were to tell you that using graphs, I could show that we're basically friends? Well, not quite – I'm a friend of a friend of a friend of a friend of a friend of a friend of yours. Confused? It's time to look at some of Erdős's work – the Erdős–Rényi model.

The Erdős–Rényi Model

One of two ways to model a random graph, the Erdős–Rényi model can be defined as;

$$G(n, p)$$

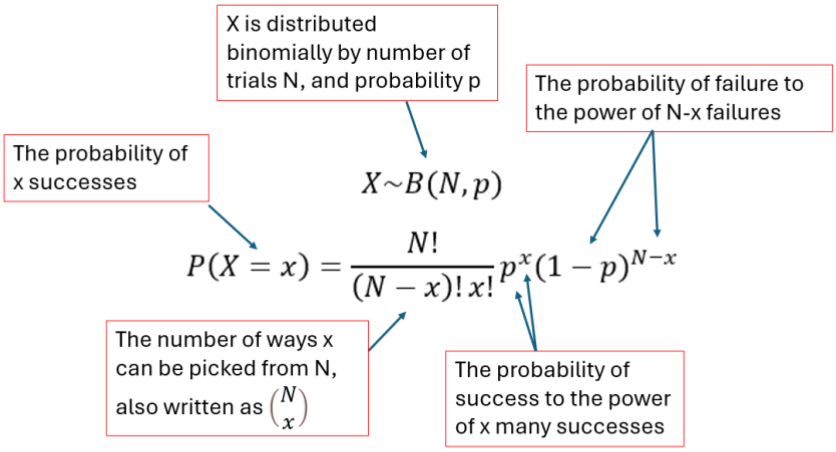
where n is the number of vertices, and a constant, independent p is the probability of an edge existing between any two vertices.

Let's consider an application of this. Say a neighbourhood has 16 cats that roam randomly around during the day, and the probability of each of them meeting in a given day is 0.2. This random graph, with cats as nodes, would be modelled as $G(16, 0.2)$. Using this model, we can work out that the most likely scenario is that a cat meets 3 others in a day – of the 15 other cats, it meets 20% of them. This doesn't mean that a cat will always meet 3 others though, that's just most likely! We can also think of this value as the most likely *degree* of a given node, as each meeting forms an edge; 3 meetings on average gives an average *degree* of 3 also.

Those familiar with the binomial distribution may notice that this scenario fits its characterising conditions:

1) Two possible outcomes of a trial (Two cats meet in a day or they don't)
2) A fixed number of trials N (A fixed number of cats to run into, $N = 15$)
3) A constant, independent probability p (A constant, independent probability of two cats meeting in a day, $p = 0.2$)

The general formula for the binomial distribution:



You can familiarise yourself easily with this by trying a few examples, before we apply this to random graphs! Try these (answers are at the end):

1) If you toss a fair coin 9 times, what's the probability of getting exactly 2 heads?
2) You try to catch 7 Pokémon with a 0.6 chance of success. What's the probability of catching all 7?
3) A cat has a 0.2 chance of meeting each of its 15 cat friends – what's the probability of it meeting exactly 6?

Great, now we're all comfortable, we can apply this formula to our random graph of $G(n, p)$:

$$K \sim B(n-1, p)$$

$$\mathrm{P}(K = k) = \binom{n-1}{k} p^k (1-p)^{(n-1)-k}$$

where k represents the degree of a node.

Using this formula, we can calculate the mean of the distribution, like we did earlier, using

$$(n-1)p = np - p$$

We can also calculate the variance, the measure of spread from the mean, using

$$(n-1)p(1-p) = np - np^2 - p + p^2$$

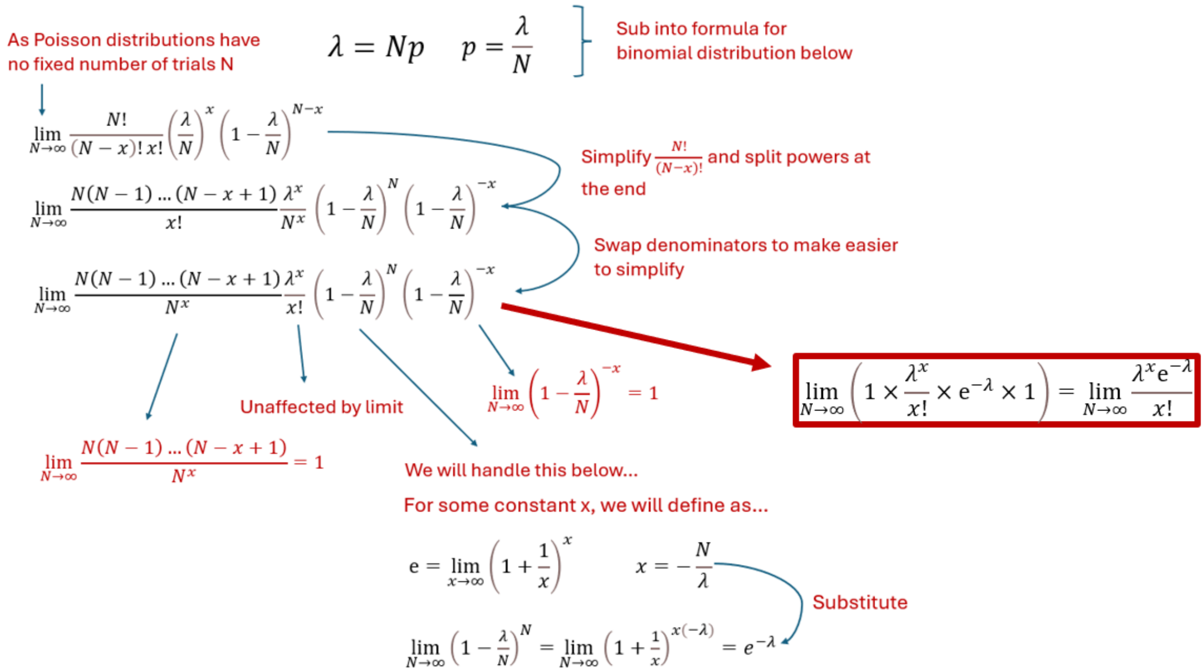
Now, in practice, it's worth noting that Erdős–Rényi graphs tend to use a large value n, and small value p. As n and p tend towards this, the mean and variance both tend towards np.

This means, for large n and small p, this distribution becomes a special, limiting case of binomial distribution – the Poisson distribution.

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

For those curious, below is a derivation of the Poisson distribution from the binomial distribution (but we only really need to know the formula!):

Note that N here refers to the N of the general binomial distribution formula, not the n of a random graph.



Poisson distributions make calculating with random graphs far easier, as they only require one parameter, λ. For our distribution, $\lambda = np$, the mean and variance. Practically, this means we can calculate the probability of a node having a certain degree, only by knowing the average degree of nodes in the graph. This is helpful – let me prove it to you using our example from earlier. Say I attached a camera to my cat's collar, so I knew that the mean number of encounters she had per day was 3. I could calculate the probability of her meeting any number of cats with only this information – I wouldn't need the fixed number of 15, or the exact probability. So, if you ever want to test your cat's chance at friendship, you know how! Let's try for her meeting 9 cats:

$$K \sim Po(3)$$

$$P(K = 9) = \frac{3^9 \times e^{-3}}{9!} = 0.00270$$

We can continue to use k to represent the degree of a node, giving us our contextualised Poisson formula of

$$P(K = k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}$$

where $\langle k \rangle$ is the expected value of k

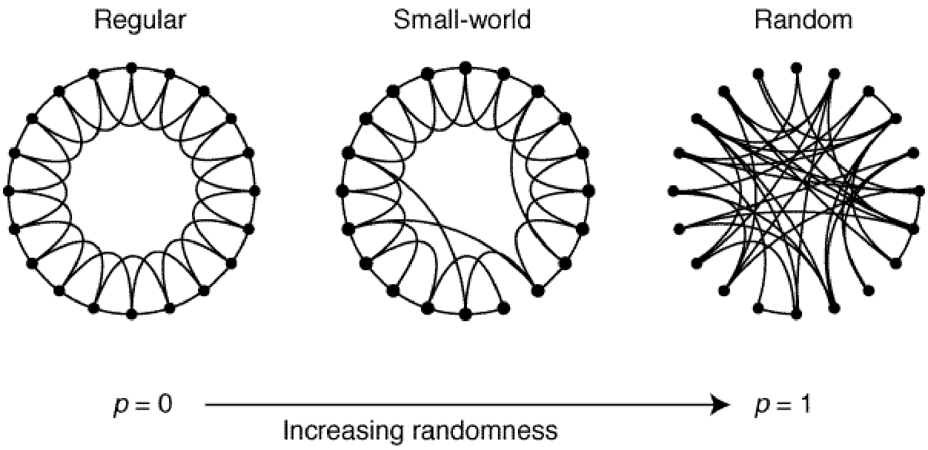
However, we still haven't hit the crux of the random graph – using only the total number of nodes, n, and the expected degree of a node, $\langle k \rangle$, we can calculate the average distance of any two nodes in any random graph. This has been shown to be, simply:

$$d = \frac{\ln(n)}{\ln(\langle k \rangle)}$$

But what's the point? What does this have to do with me being only six people away from you? The world isn't random – I don't have friends from every country in the world, of any age demographic, any interests. I have friends that meet certain criteria, and I'm sure you do too. My world is small! So, two men came up with a solution in the 1990s to model this: introducing the Watts-Strogatz *Small World Network*.

Small World Networks: Back to Reality

The key principle here is that people form social circles (aptly named) where friendships often overlap, rather than being randomly allocated – it's not equally as likely that I'm friends with Beyoncé as it is that I am with Betty from next door! Therefore, Watts and Strogatz started their model using a regular lattice (a graph where all nodes have the same degree) and used a parameter p to determine the probability of a node being 'rewired' to another node in the lattice, essentially forming a half-way point between order and total chaos. This parameter p ranged $0 \leq p \leq 1$, with $p = 0$ forming a totally regular lattice, and $p = 1$ forming a random graph.



This simulates real-life one-off meetings; someone you sat next to on a plane, the barista you befriended while staying at a relative's house, and so forth. This rewiring connects distant groups within the lattice, making the distance between any two nodes smaller as there's an alternative path to take.

What Watts and Strogatz noticed however, was that not that many nodes needed to be 'rewired' before their model approached the properties of a totally random graph. They called this model a 'Small World Network' – because after all, we've all had that moment where our worlds collide!

This means that the formula for the average distance in a random graph from earlier,

$$d = \frac{\ln(n)}{\ln(\langle k \rangle)}$$

can apply here too, and will finally prove that odds are, we're basically friends.

Global population is currently about 8.2 billion, so discounting young children, the very elderly, and those with unusual social patterns, lets call $n = 7{,}000{,}000{,}000$. I'm going to use an estimate of about 35 friends per person, accounting for colleagues and family also – so $\langle k \rangle = 35$. Here we go…

$$\frac{\ln(7{,}000{,}000{,}000)}{\ln(35)} = 6.38\ (3sf)$$

So, the average distance between any two people (exhibiting normal social patterns) is only 6.38, using a Small World Model! Okay, benefit of the doubt – maybe we're 7 people apart not 6…

In Our Lives Now

This mathematical evidence backs up Stanley Milgram's social experiment from the 1960s, where he chose a selection of people in Nebraska to pass letters along to targets in Boston, coining his theory of 'six degrees of separation' that's achieved global attention in recent years. It was then adapted by Watts and Strogatz themselves, who conducted a version of the experiment using social media, where they used X and Facebook, giving average path lengths of 3.4 and 4.5 respectively, proving the sheer impact of social media in connecting us. I'll leave the substitution of the median Facebook followers of 155 into the formula as an exercise to the reader – the results are startlingly accurate!

What a journey! We've travelled from the most social mathematician of all time straight into his work, to explain an inside joke from his friends. We've met graphs using Pokémon and learnt the key properties. We've explored the binomial distribution, derived the Poisson distribution from that, then ventured into using logarithms to find average path length. We've discovered that everyone is friends with everyone if you try hard enough… and that I'm probably the least social mathematician of all time, spending my weekend writing this essay! So maybe we would be friends after all, as you've spent yours reading this – we've got to stick together.

Answers

1. P(X=x) = 0.0703 (3sf)
2. P(X=x) = 0.0280 (3sf)
3. P(X=x) = 0.0430 (3sf)

Acknowledgements

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