0.5! and the Gamma Function

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1. The factorial and its limitations

Our journey begins with a simple problem:

I have 5 distinct letters to choose from – A, B, C, D and E. In how many distinct ways can I arrange them? I have 5 'slots' for one letter each – the first can have any of the five letters, so there are 5 possibilities. The second slot now has one less letter in our option pool as we just used one, so there are only 4. The third slot has 3, the fourth has 2 and we only have 1 for our last slot. We then multiply all of these numbers together to find our final answer:

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

This calculation that we did – multiplying all the natural numbers up to our number – was given a name: the factorial function, denoted as $n!$. It reportedly first appeared in Indian scripts as well as in the Middle East, where work was supposedly done on the number of words that could be formed from the Hebrew alphabet. It is now a powerful tool in combinatorics and number theory – in this essay, we will observe its uses in analysis. A useful result from the factorial is that $n! = n \times (n-1)!$. This will prove useful later.

There is one flaw with the factorial function: take $n = 4.5$. The factorial is defined for natural n as:

$$n(n-1)(n-2)(n-3) \ldots 2 \times 1 = \prod_{k=1}^{n} k$$

However, following this would result in us getting:

$$4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times -0.5 \times -1.5 \ldots$$

We should stop multiplying when we get to 1, but we never reach it. Hence, the factorial is not defined for all the real numbers, limiting its uses in analysis. Mathematicians now needed to find a way to represent the factorial with a continuous function. Fortunately, Swiss mathematician Leonhard Euler published a groundbreaking work in the 18^{th} century that solved this problem, and this work would later be developed by mathematicians such as Legendre, Weierstrass, Riemann and more!

2. The Gamma Function

2.1 The definition

Consider $f(x) = x^n$. Using the power rule of differentiation, we get $f'(x) = nx^{n-1}, f''(x) = n(n-1)x^{n-2}, f'''(x) = n(n-1)(n-2)x^{n-3}$ and so on. A pattern emerges from this – each time we differentiate, we multiply by the current power and then subtract one from it. We do this until we get to a power of one. This bears semblance to our definition of the factorial. In fact:

$$\forall n \in \mathbb{N}, x \in \mathbb{R}, f(x) = x^n \Rightarrow f^{(n)}(x) = n!$$

The x disappears (as we get $x^{n-n} = x^0 = 1$) and we are left with a factorial. This yields an integral that is equivalent to the factorial.

However, integration is the opposite of differentiation – can we differentiate x^n if it is inside an integral?

Consider the product rule of differentiation:

$$(fg)' = f'g + g'f$$

Taking the indefinite integral on both sides and rearranging this, we get:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

This is the well-known integration by parts (IBP) formula. As those who have used it know, one must choose a function to differentiate and one to integrate. Here, Euler chose to repeatedly differentiate x^n. What did he choose to integrate?

The Gamma function is defined as follows:

$$\Gamma(z) = \int_0^\infty x^{z-1}\,e^{-x}dx$$

Integrating by parts (differentiating x^{z-1} and integrating $\exp(-x)$):

$$-x^{z-1}e^{-x}]_0^\infty - (-)(z-1)\int_0^\infty x^{z-2}e^{-x}dx = (z-1)\int_0^\infty x^{z-2}e^{-x}dx$$

We now see that Euler used the exponential function because the calculation outside the integral becomes 0, and the integral of the exponential function remains basically the same (except for the factor of -1, which is used for issues of convergence and the sign of our answer). We apply IBP again to get:

$$(z-1)(z-2)\int_0^\infty x^{z-3}e^{-x}dx$$

Finally, we eventually get:

$$(z-1)(z-2)(z-3)\dots 1\int_0^\infty e^{-x}dx$$

The integral on the right evaluates to 1, so as one can see:

$$\forall z \in \mathbb{N}, \Gamma(\mathrm{z}) = (\mathrm{z}-1)!$$

The Gamma function also has the property that $\Gamma(\mathrm{z}+1) = \mathrm{z}\Gamma(\mathrm{z})$. This is analogous to $n! = n(n-1)!$. Though it can be seen as just a factorial, it has many benefits over the normal factorial, namely:

- It is continuous and differentiable.
- It is defined for all real numbers except non-positive integers (more on that later).
- It also takes complex numbers as arguments (except where it has simple poles/is undefined, where the real part is a non-positive integer).

This makes this function powerful in many areas of mathematics, such as complex analysis, statistics and quantum mechanics. Let us now discuss some properties of this function.

2.2 Some properties of the Gamma Function

Firstly, as previously mentioned:

$$\Gamma(\mathrm{z}+1) = \mathrm{z}\Gamma(\mathrm{z})$$

There is one major flaw with the Gamma Function – though it is defined for real and complex numbers, it is not defined for non-positive integers (this includes 0), e.g. $0, -1, -2, -3$, etc.

$$\Gamma(\mathrm{z}+1) = \mathrm{z}\Gamma(z) \Leftrightarrow \Gamma(z) = \frac{\Gamma(z+1)}{z}$$

Consider $\Gamma(0)$:

$$\Gamma(0) = \frac{\Gamma(1)}{0}$$

We can’t divide by 0, though, so this doesn’t exist.

$$\Gamma(-1) = \frac{\Gamma(0)}{-1}$$

As $\Gamma(0)$ doesn't exist, neither does $\Gamma(-1)$. Then $\Gamma(-2)$, $\Gamma(-3)$, $\Gamma(-4)$ and so on all don't exist in this fashion.

3. The Generalised Gaussian Integral

At the start of this section, we will deviate from our study of the Gamma function for some time. However, we will loop it back in in the final part.

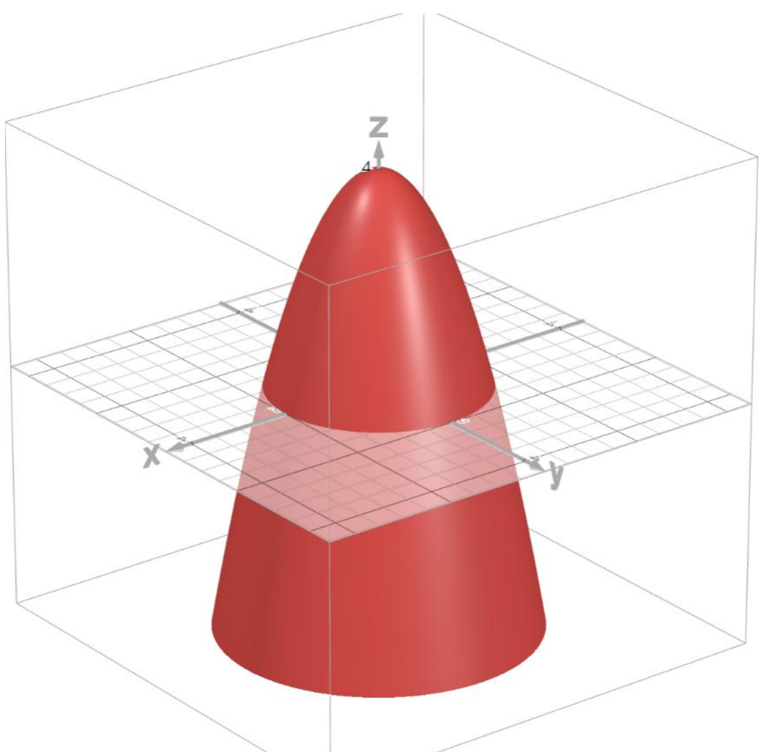
3.1 A brief introduction to Double Integrals and Polar Coordinates

Single-variable integration is the study of areas under curves in 2-D space. An integral is defined by a Riemann Sum as follows:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

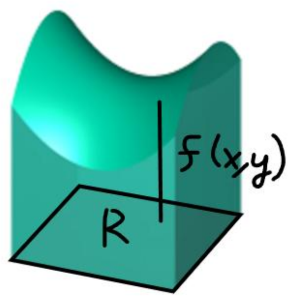
This is taking the sum of rectangles of width $\frac{b-a}{n}$ and height $f(x)$, then making the width become infinitely small or tend to 0. Hence, the integral is just the limit of a sum. How do we extend this to 3 dimensions?

Equation 1: $f(x, y) = 4 - x^2 - y^2$ (Taken from desmos.com)

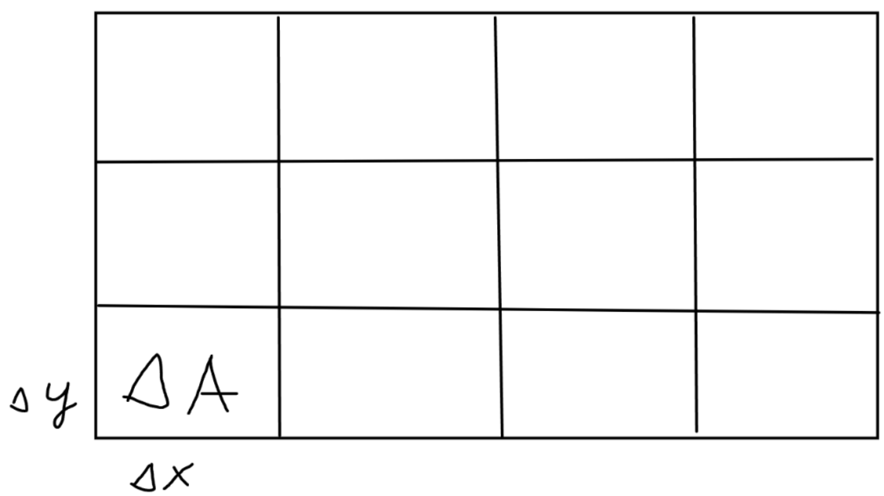


Shown in Equation 1 is a 3D surface – a function of two variables, not one. Hence, we wouldn't be finding areas under the surface – we would be finding the volumes under it.

Intuitively, an integral takes us from length to area, so an integral of an integral (an iterated integral) should take us from area to volume. Instead of integrating over a length under a curve, we integrate over a region under a surface (in this specific case, we will use a rectangular region (x and y vary between 2 real constants each) for our change of variables). Consider the following image:



The volume of a prism is the cross-sectional area multiplied by the height. Let's zoom into R:



Our cross-sectional area is ΔA. We can see that $\Delta A = \Delta x \Delta y$. Hence, we just take the sum of the cross-sectional areas multiplied by the height, $f(x, y)$. This yields:

$$\sum f(x,y)\Delta A = \sum f(x,y)\Delta x \Delta y$$

This idea is what we are going to use to (informally) derive a version of Fubini's theorem, published by Guido Fubini in 1907.

A corollary of Fubini's Theorem states that:

$$\left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right) = \int_c^d \int_a^b f(x)g(y)\,dx\,dy$$

Well, we can write the 2 integrals on the LHS as follows:

$$\left(\lim_{m\to\infty}\sum_{i=1}^{m} f(x_i)\Delta x\right)\left(\lim_{n\to\infty}\sum_{j=1}^{n} g(y_j)\,\Delta y\right)$$

Where $x_i = a + i\Delta x$ and $\Delta x = \frac{b-a}{m}$ (this is defined similarly for y).
Let $p_i = f(x_i)\Delta x$ and $q_j = g(y_j)\Delta y$. Then, we get:

$$\lim_{m\to\infty}(p_1 + p_2 + p_3 \ldots + p_m) \times \lim_{n\to\infty}(q_1 + q_2 + q_3 \ldots + q_n)$$
$$= \lim_{(m,n)\to(\infty,\infty)}(p_1 + p_2 + p_3 \ldots + p_m)(q_1 + q_2 + q_3 \ldots + q_n)$$

Only considering the operand inside the limit, we see that:

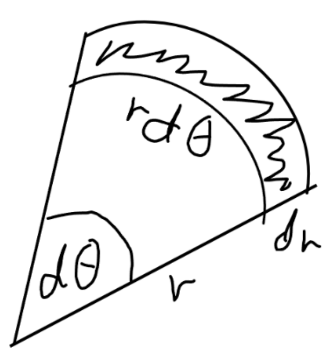
$$(p_1 + p_2 + p_3 \ldots + p_m)(q_1 + q_2 + q_3 \ldots + q_n)$$
$$= p_1(q_1 + q_2 + q_3 \ldots + q_n) + p_2(q_1 + q_2 + q_3 \ldots + q_n) \ldots$$
$$+ p_m(q_1 + q_2 + q_3 \ldots + q_n) = \sum_{i=1}^{m}\left(p_i \sum_{j=1}^{n} q_j\right)$$

$$\lim_{(m,n)\to(\infty,\infty)}\left(\sum_{i=1}^{m}\left(p_i \sum_{j=1}^{n} q_j\right)\right) = \lim_{(m,n)\to(\infty,\infty)}\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n} f(x_i)g(y_j)\Delta y\Delta x\right)\right)$$
$$= \int_a^b \int_c^d f(x)g(y)\,dy\,dx$$

We can move p_i into the sum from $j = 1$ to n as it is independent of j.

We have now proven our first result needed to understand a famous solution to the Gaussian Integral.

Our second result needed is related to changes of variables with multiple integrals. If we make the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, we move to polar coordinates, where the x and y coordinates form a right-angled triangle in a circle of radius r with centre $(0,0)$ because $x^2 + y^2 = r^2$. The rectangles that we used earlier for our area differential dA now become sectors of this circle:



Observe the (poorly!) shaded region – a ‘polar rectangle.’ It is curved, but we can still find its area – it is the arc length multiplied by dr. The arc length is just the radius multiplied by our change in our angle – $rd\theta$. Hence, in this case, $dA = rd\theta \times dr = rdrd\theta$.

We are now equipped with the tools to tackle the famous Gaussian Integral.

3.2 The Gaussian Integral (of degree 2)

The Gaussian Integral is defined as follows:

$$I = \int_0^\infty e^{-x^2}\,dx$$

Normal u-substitution doesn’t work here, so we transform it into a double integral.

Consider I^2:

$$\left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-x^2}\,dx\right)$$

If we set $x = y$ in the second integral, $dx = dy$, yielding:

$$\left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

According to Fubini’s theorem, this is:

$$\left(\int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}dxdy\right) = \left(\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy\right)$$

Moving to polar coordinates:

$$x = r\cos\theta\,, y = r\sin\theta$$
$$dA = rdrd\theta$$

The region $[0,\infty)^2$ describes the 1st quadrant of the cartesian plane. This is at an angle of $\frac{\pi}{2}$ radians anti-clockwise from the x-axis. The radius also stretches out to infinity. Hence, $0 \le r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$.

We can replace our bounds in the double integral in this way:

$$\left(\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta\right)$$
$$= -\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \int_0^\infty -2re^{-r^2}drd\theta\right)$$

This is now a relatively easy integral to solve. Set $u = -r^2$. Then $\frac{du}{dr} = -2r$, so $\frac{du}{-2r} = dr$. As $r \to \infty$, $u \to -\infty$. When $r = 0$, $u = 0$.

$$-\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\int_0^{-\infty} -2re^u \frac{du}{-2r} d\theta\right)$$

$$= -\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\int_0^{-\infty} e^u du d\theta\right)$$

$$= -\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} e^u]_0^{-\infty} d\theta\right)$$

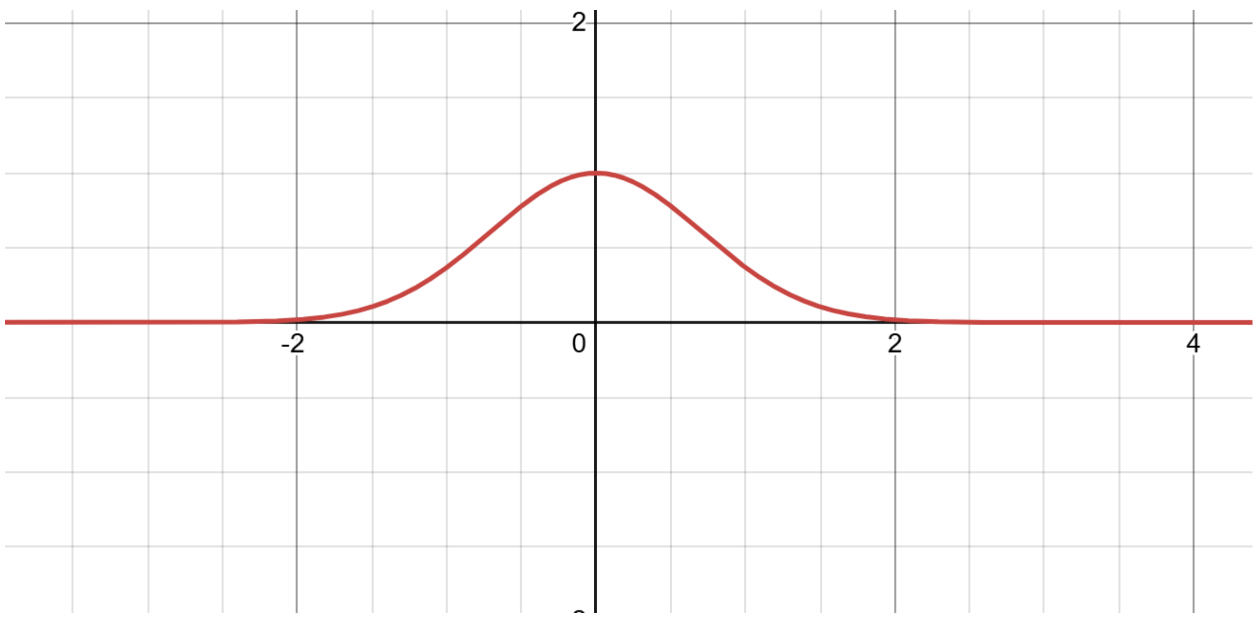
$$= -\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} -1 d\theta\right)$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

Note that e^{-x^2} is positive for all real x, so $\int_0^\infty e^{-x^2} dx$ must also be positive. Hence $I > 0$.

Equation 2: $y = e^{-x^2}$



Finally:

$$I^2 = \frac{\pi}{4}, I > 0$$

$$\therefore I = \frac{\sqrt{\pi}}{2}$$

We will now try to relate the integral over the positive real numbers of e^{-x^n}, where n is a positive integer, to the Gamma function.

3.3 The GGI (Generalised Gaussian Integral) and the Gamma Function

Consider $\int_0^\infty e^{-x^n} dx$. Let $u = x^n \Leftrightarrow x = u^{\frac{1}{n}}$. Then $\frac{du}{nx^{n-1}} = dx.\ x^{n-1} = \frac{x^n}{x} = \frac{u}{u^{\frac{1}{n}}} = u^{1-\frac{1}{n}}$.

Hence $u^{-(1-\frac{1}{n})} du = dx \Leftrightarrow u^{\frac{1}{n}-1} du = dx$. As $x \to \infty, u \to \infty$ and when $x = 0$, $u = 0$ as well.

Now:

$$\int_0^\infty e^{-x^n} dx = \frac{1}{n} \int_0^\infty u^{\frac{1}{n}-1} e^{-u} du$$

However, the definition of the Gamma Function is:

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du$$

By inspection, one can see that $z = \frac{1}{n}$ in this case ($u^{\frac{1}{n}-1} = u^{z-1} \Leftrightarrow z = \frac{1}{n}$).

$$\int_0^\infty e^{-x^n} dx = \frac{1}{n} \Gamma\left(\frac{1}{n}\right)$$

Using $\Gamma(n+1) = n\Gamma(n)$:

$$\frac{1}{n}\Gamma\left(\frac{1}{n}\right) = \Gamma\left(\frac{1}{n} + 1\right) = \Gamma(\frac{n+1}{n})$$

$$\therefore \forall n \in \mathbb{Z}^+, \qquad \int_0^\infty e^{-x^n} dx = \Gamma(\frac{n+1}{n})$$

This is a powerful result that we will now use to find the half-integer values of the Gamma Function.

4. The half-integer values of the Gamma Function, generalised

4.1 Finding ½ Factorial

Using the result from the previous section:

$$\int_0^\infty e^{-x^n} dx = \Gamma(\frac{n+1}{n})$$

We can plug in $n = 2$ to see that:

$$\int_0^\infty e^{-x^2} dx = \Gamma\left(\frac{3}{2}\right)$$

But we also know that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$\Gamma(n) = (n-1)!$ yielding:

$$\frac{1}{2}! = 0.5! = \frac{\sqrt{\pi}}{2}$$

This answers our initial question. Let us try to generalize this result for all multiples of $\frac{1}{2}$ or the half-integers.

4.2 Generalisation to the half-integers

Now, we want to find the value of the gamma function for the nth half-integer $\frac{2n+1}{2}$. We can do this by observing the recurrence relation that we discussed in section 2.2 ($\Gamma(z+1) = z\Gamma(z)$). Applying this here, we get:

$$\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$\therefore \Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

For $z = \frac{1}{2}$:

$$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{2}$$

$$\frac{\sqrt{\pi}}{2} = \frac{\Gamma\left(\frac{1}{2}\right)}{2}$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

How do we find a formula for $\Gamma\left(\frac{2n+1}{2}\right)$?

Well, if we start at $n = 0$, we get $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. If we move up to $n = 1$, we get $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$. If we move to $n = 2$, we get $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3\times1}{2\times2}\Gamma\left(\frac{1}{2}\right)$. For $n = 3$, we get $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5\times3\times1}{2\times2\times2}\Gamma\left(\frac{1}{2}\right)$. For any half integer $z = \frac{2n+1}{2}$, we get $\Gamma(z) = \frac{(2n-1)(2n-3)(2n-5)\ldots.\times5\times3\times1}{2^n}\Gamma\left(\frac{1}{2}\right)$. This is the product of all the odd numbers before z multiplied by $\Gamma\left(\frac{1}{2}\right) \times \frac{1}{2}$, or $\frac{\sqrt{\pi}}{2}$. This looks like a factorial, but it isn't the same – a factorial is a product of consecutive integers, while this is the product of consecutive *odd* integers. How do we express this?

With $k!$, our step down is by 1, e.g. $k(k-1)(k-2)\ldots \times 2 \times 1$. There exists another function, known as the double factorial (written $k!!$), where the step down is 2 $k!! = k(k-2)(k-4)(k-6)\ldots$

Hence, for the positive half integers:

$$\forall n \in \mathbb{Z}_0^+, \qquad \Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}$$

But what about negative half-integers? Well:

$$\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

So:

$$\Gamma\left(-\frac{1}{2}\right) = -2 \times \Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

Then:

$$-\frac{3}{2}\Gamma\left(-\frac{3}{2}\right) = \Gamma\left(-\frac{3}{2}+1\right) = \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$\therefore \Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3} \times -2\sqrt{\pi} = \frac{4\sqrt{\pi}}{3}$$

Here, we can see that the coefficients of $\sqrt{\pi}$ are simply the reciprocals of their positive counterparts, with an extra factor of $(-1)^n$ as the coefficient's sign alternates between positive and negative. If n is negative, then we can make n positive with $|n|$, the modulus function.If $\Gamma\left(\frac{2n+1}{2}\right) = k\sqrt{\pi}$ then $k = \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\sqrt{\pi}}$. Hence, we get:

$$\frac{(-1)^n\sqrt{\pi}}{k} = \frac{(-1)^n\sqrt{\pi}}{\frac{\Gamma\left(\frac{2|n|+1}{2^{|n|}}\right)}{\sqrt{\pi}}} = \frac{(-1)^n \times \sqrt{\pi} \times \sqrt{\pi}}{\frac{(2|n|-1)!!}{2^{|n|}}\sqrt{\pi}} = \frac{(-1)^n 2^{|n|}\sqrt{\pi}}{(2|n|-1)!!}$$

Finally, we get 2 equalities for the half integer values of the Gamma Function:

$$\forall n \in \mathbb{Z}_0^+, \qquad \Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!\,\sqrt{\pi}}{2^n}$$

$$\forall n \in \mathbb{Z}^-, \qquad \Gamma\left(\frac{2n+1}{2}\right) = \frac{(-1)^n 2^{|n|}\sqrt{\pi}}{(2|n|-1)!!}$$

5. Conclusion

Well, we have now reached the end of our breathtaking mathematical journey. We introduced the Gamma Function and its properties and found the answer to our initial question. Moreover, we found a relationship between the Gamma Function and the GGI:

$$\int_0^\infty e^{-x^n} dx = \Gamma(\frac{n+1}{n})$$

Furthermore, we found general formulae for the values of the Gamma function at half-integers:

$$\forall n \in \mathbb{Z}_0^+, \qquad \Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2}$$

$$\forall n \in \mathbb{Z}^-, \qquad \Gamma\left(\frac{2n+1}{2}\right) = \frac{(-1)^n 2\sqrt{\pi}}{(2|n|-1)!!}$$

To the reader: I hope you have enjoyed this journey centered around such a beautiful piece of mathematics as I have. It really shows how powerful it can be when you take an idea and extend it to larger varieties of situations. These results prove essential when tackling many different integrals, such as the Bose-Einstein integral in Quantum Mechanics, the Laplace transform of power functions, and other mathematical problems. I hope that you learned something new from this essay and my final message will be to never stop learning – a deep passion for mathematics is what allowed Euler, Laplace and other legendary mathematicians to revolutionise the field as they have.