A Babble about Basel: the underlying unity within mathematics

By: Leo Hill

Introduction:

To set the scene: the Basel problem, deceptively difficult, was first proposed by Pietro Mengoli in 1650, the summation of the reciprocals of the squares of the natural numbers. After many years of countless mathematicians losing to this fierce opposition, including the astonishing Bernouli's, no one had yet to find an answer to this apparent "impossible" question.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

However, nearly a century later a solution had been found and it had taken a 26 year old genius, Leonhard Euler, to approach this problem from a remarkable way, generating a striking answer within the original settings of the question, creating a link between series and π:

$$\frac{\pi^2}{6}$$

Following Euler's breakthrough, the flood gates opened. A barrage of differing methods emerged - from many areas of mathematics – aimed to find a solution this once "unanswerable" problem.

At my school's Turing Maths club, our discussions had been focussed around series and summations this term culminating on a finishing chapter of Euler's elegant proof to the Basel problem. Inspired by this, I found that there was not only Euler's proof but rather his was one of many.

In this essay, I will draw upon multiple mathematical fields all used in an approach to find a solution to Mengoli's problem – granting an insight into the interconnectedness of mathematical ideas and brilliant mathematic minds.

Euler's original proof:

Euler, determined to find an answer to this question, begin by considering the sine function which to many seems to have no link to the original problem.

Step 1: Roots of polynomials

For any polynomial $P(x)$, $P(x)$ can be written as a product of its roots:

$$P(x) = a(x - a_1)(x - a_2) \dots (x - a_n)$$

Euler observed that the sine function had roots at: $x = n\pi$ (for all integer values n). With this, Euler took a leap to rewrite $\sin x$ analogous to an infinite degree polynomial:

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

Dividing through by x gives:

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

Euler used this product representation heuristically, without a full rigorous justification. This was only later made precise in the 19th century.

Step 2: Utilising the Maclaurin series expansion of $\sin x$

The Maclaurin series expresses a function as an infinite polynomial determined by its derivatives at $x = 0$, providing an approximation which is exact under suitable conditions.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

There is a more general method of polynomial approximation known as the Taylor series which represents a function as an infinite sum of powers about $x = a$ and this is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

However, due to the sine function being focussed around $x = 0$ we can use the Maclaurin series here, which for $\sin x$ is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

Similarly to step 1, we can also divide through by x here:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots$$

Step 3: Delving further into the product expansion

Considering:

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

When expanded, the x^2 term's coefficient comes from summing the first-order terms for each factor as selecting the linear term from one factor and constant terms from all others produces all the contributions to the coefficient:

$$-\left(\frac{1}{1^2\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} \ldots\right)$$

Therefore, we can rewrite the start of the infinite product as:

$$1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\right) x^2 \ldots$$

Step 4: Equating x^2 coefficients

The Maclaurin series of the sine function had been known for decades prior to Euler's proof. However, the ingenuity of Euler's solution arises from his key insight to equate both coefficients, giving him an answer to this near century old problem. This is because each representation describes the same function and therefore, their power series expansion must agree on each individual term.

From the product expansion, the coefficient of the x^2 term:

$$-\left(\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\right)$$

From the Maclaurin expansion, the coefficient of the x^2 term:

$$-\frac{1}{6}$$

Thus equating these grants:

$$-\left(\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\right)=-\frac{1}{6}$$

Since the π^2 is constant in the summation, it can be factored out, and additionally multiplying through by -1 thus giving:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

An elegant solution had been found by a surprising crosslink between the geometry of π and infinite series. Euler's insight, which led this unexpected result, remains as one of the most remarkable achievements in 18th century mathematics.

The Fourier Series proof:

This next proof draws upon another field of mathematics which similarly to Euler has seemingly nothing to do with the Basel problem. However, dissimilarly to Euler, this proof is a bit more complicated than the original and involves calculus!

A Fourier Series is a way to express a function as an infinite sum of sines and cosines. For a function $f(x)$, defined on an interval, it is given by:

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$$

This allows for analysis of complicated function through a trigonometric series, which for some functions is otherwise much harder to do.

However, just given this formula, Euler's original answer looks out of reach. Therefore, Parseval's identity can be used to relate the Fourier coefficients to the integral of its square *(quite longwinded)*.

Parseval's identity is given for a function $f(x)$ bounded by $[-\pi,\pi]$ which in this case is odd:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

This may look like a very complicated formula, however, we can think of the left hand side being the total "energy" of the function and the right hand side being the sum of the "energies" of its Fourier components. Therefore, we can essentially view Parseval's identity as a conservation of energy principle but for functions.

Armed with Parseval's identity, we can begin to build a proof for the Basel problem.

Step 1: Select suitable function

To make things easier, we should aim to remove as many constants and variables as possible. An odd function is one which satisfies:

$$f(-x) = -f(x)$$

Which is useful in this situation as we know that and odd function multiplied by an even is odd, and as such to compute the answer we should choose an odd function.

Occam's razor, although intrinsically tied to philosophy but somewhat relevant in this situation, states that "plurality should not be posited without necessity," which made easier to understand means given two arguments for the same outcome, the simpler one is more proficient.

Therefore, when selecting our function we should aim to make it both odd and as simple as possible, which given these constraints leads to $f(x) = x$

Step 2: Find the function's Fourier series

Since, $f(x)$ is odd, all cosine coefficients within the Fourier series = 0 $(a_n = 0)$.

Therefore, only the sine terms remain, so the series must resemble:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx)$$

We also know that due to the orthogonality of sine and cosine:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Which for $f(x) = x$:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

As mentioned previously, $f(x)$ is an odd function, as is $\sin(nx)$. In this situation, given that an odd function multiplied by another odd function is equal to an even function, we can use symmetry to generate:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

To compute this, use integration by parts:

$$u = x \qquad du = dx$$

$$dv = \sin(nx) \qquad v = -\frac{\cos(nx)}{n}$$

So:

$$\int_0^{\pi} x\sin(nx)\,dx = -\frac{x\cos(nx)}{n} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx$$

$$\int_0^{\pi} x\sin(nx)\,dx = \frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}$$

Now evaluating the bounds and cancelling:

$$\int_0^{\pi} x\sin(nx)\,dx = \frac{-\pi(-1)^n}{n}$$

Substituting back for b_n:

$$b_n = \frac{2}{\pi} \times \frac{-\pi(-1)^n}{n}$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

Which again substituting:

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$$

This step looks very complicated, however, a year of A-level maths helps!

Step 3: Applying Parseval's identity

Extending $f(x) = x$ as an odd function to [$-\pi, \pi$] ensures all cosine coefficients = 0, therefore Parseval's identity is a sine-series form in this interval.

$$\int_0^{\pi} x^2\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} b_n^2$$

Not dissimilar to Euler, the answer to the actual question looks to appear from thin air without much mention to it throughout the proof.

Computing the left hand side with simple integration and substituting b_n, we get that:

$$\frac{\pi^3}{3} = \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2$$

Which simplifies to:

$$\frac{\pi^3}{3} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{4}{n^2}$$

Since:

$$\sum_{n=1}^{\infty}\frac{4}{n^2} = 4\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$$

This means that the equation can simplify to:

$$\frac{\pi^3}{3} = 2\pi\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Which by dividing through by 2π grants us again that:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

While this proof is not as intuitive/beginner friendly as Euler's original, the fundamental answer remains the same.

Conclusion:

Having only highlighted two proofs of the Basel problem in this essay, the apparent unity within mathematics may not easily come to mind. However, while Euler's proof relied on differing ways of rewriting a trigonometric function the Fourier approach follows a path of harmonic decomposition and integration, yet both converge to the same point:

$$\frac{\pi^2}{6}$$

Therefore, it is easy to see the Basel problem and remark at the ingenuity of Euler. However I believe that there is a deeper message: it demonstrates how mathematical ideas are interconnected across seemingly unrelated fields, and how differing methods can still ultimately lead to the same result.

References:

Euler, L. (1735) *De summis serierum reciprocarum*. (Original work on the Basel problem)

"Basel problem", Wikipedia https://en.wikipedia.org/wiki/Basel_problem

Bazett, T. 2022 Parseval's Identity, Fourier Series, and Solving this Classic Pi Formula