A Day at the Beach

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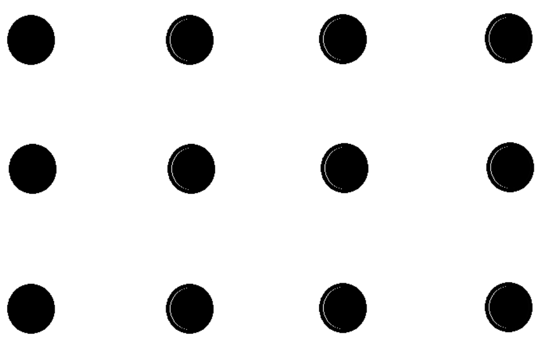
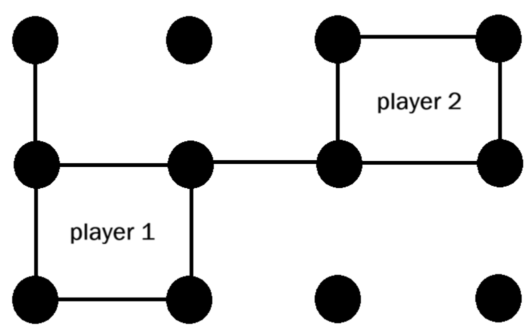
La Bocca is a nice beach in Cannes, southern France. Today, we'll go there and hopefully have some fun!

Alright, the bus has just stopped, let's get our toes in the sand. Wow! The sky is so blue; the sea as well, as if the sky became one with the sea. Anyway, we are a bit full after our heavy breakfast. Maybe we should wait a few hours before swimming. To kill time, let's play in the sand!

Time to play our favorite game: "dots and boxes"! Let's play with our fingers in the sand. A recap for the newcomer: to play "dots and boxes", we start by drawing some dots in a grid-like way; then connect two adjacent dots by vertical or horizontal lines alternating turns; as soon as a player forms a square, they score one point. The player who forms the most squares wins the game.



A Blank 3 x 4 Game

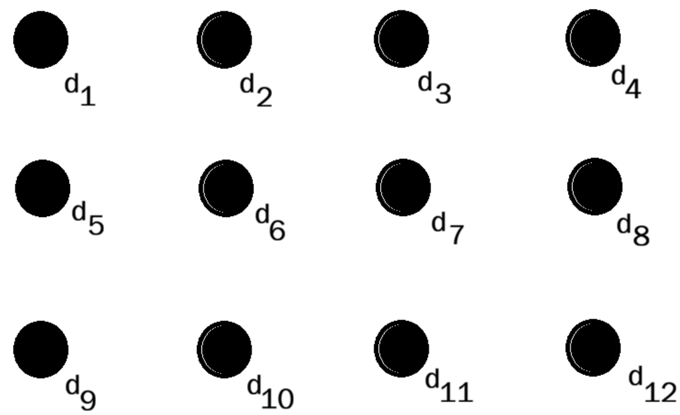


A State of a 3 x 4 Game

Fun fact: "dots and boxes" was popularised by the French mathematician, Édouard Lucas who called the game *La Pipopipette*.

Alright, let's be serious for a moment and recall that this is an essay on a mathematical topic.

The "dots" on the sand can be labelled as $d_1, d_2, d_3, \dots, d_{12}$:



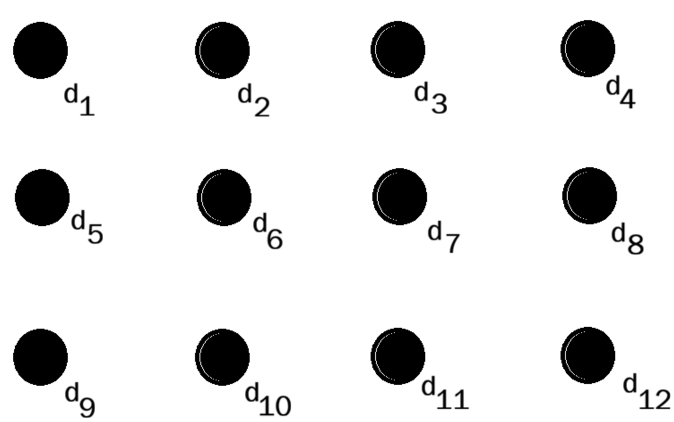
Labelled Dots

Now, let's go a step further and put those "dots" into a "bag". What types of bags have we learnt since primary school? That's right: sets. So, let's put our dots into a set D such that:

$$D = \{d_1, d_2, d_3, \dots, d_{12}\}$$

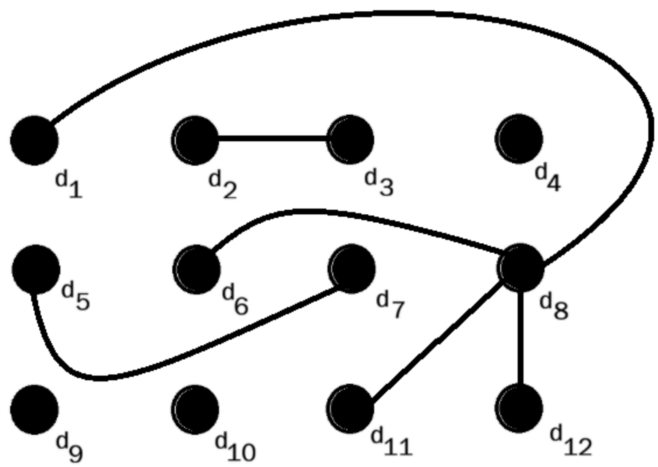
Okay, it looks like we have a kind of structure for representing our "dots". Let's now go a step further and try to represent the "lines" drawn in the sand.

Let's consider a blank sand board:



Blank Board

For a moment, let's ignore the restrictions imposed by "dots and boxes" (namely, any line has to be either horizontal or vertical, and needs to connect any pair of dots that are adjacent). So, in this setting, the connections shown in the diagram below are allowed:



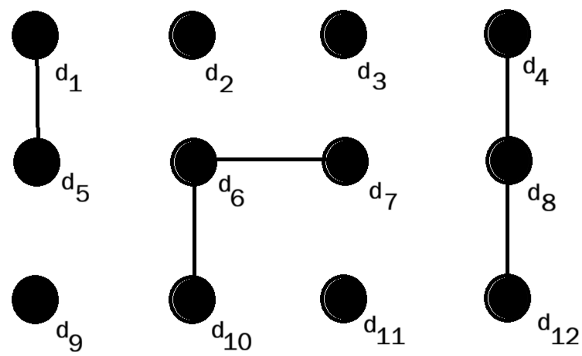
"Dots and Boxes" Without Its Adjacency Condition

In the above diagram, to form a line connecting any two dots, we simply have to pick any two dots $d_i, d_j \in D$ such that $i \neq j$. Therefore, we end up selecting a tuple or pair (d_i, d_j) where $i \neq j$. So, to obtain the set of all possible lines (including those not limited by the rules of "dots and boxes"), we need to select all such tuples. What does this remind us of? The Cartesian product of $D \times D$ of course!

Now, let's return to "dots and boxes" and impose its conditions (each line is either horizontal or vertical, and connects adjacent dots). To obtain the set of lines L , we know that L must be a subset of $D \times D$.

Having obtained mathematical representations for the set of dots ($D = \{d_1, d_2, d_3, \dots , d_{12}\}$) and the set of lines ($L \subseteq D \times D$), it becomes easy to represent any game instance of "dots and boxes".

As an example:



“Dots and Boxes” With Its Adjacency Condition

The above game could be represented with:

$$D = \{d_1, d_2, d_3, \dots , d_{12}\} \text{ and } L = \{(d_1, d_5), (d_6, d_7), (d_6, d_{10}), (d_4, d_8), (d_8, d_{12})\}.$$

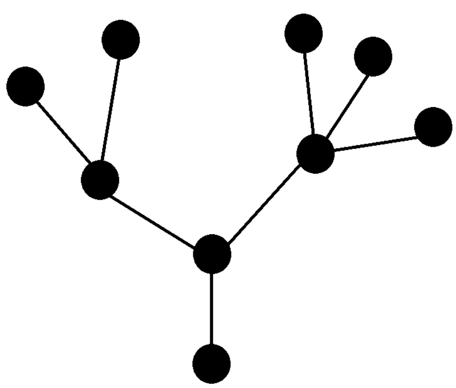
Note: *the technique that we have just used (that is to say, relaxing the conditions of a problem, obtaining a raw solution, and then reintroducing the conditions to refine our raw solution) is a common tool in the scientist’s toolkit. This method is formally termed “idealisation followed by de-idealisation”.*

Should we leave it here? After all, who likes mathematics? Nevermind. Our next move would be to create a tuple (D, L). Once combined, such a tuple is able to describe any set of dots and the set of lines connecting them.

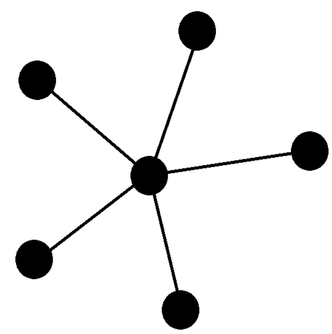
Alright, we have now obtained a generalisation for such structures. Unfortunately, we have not discovered anything new. This tuple (D, L) has already been known for more than 200 years. The tuple has a name. It is called a “graph”, which can be quite confusing at first sight. That is because by “graph”, we do not mean mathematical curves in the Cartesian plane, the ones to which we are accustomed (such as quadratic or sinusoidal ones). Rather, a “graph” is the kind of structure that emerges collectively from a set of dots and a set of lines.

Anyway, let’s get an ice cream, it is getting too hot. Mmm . . . strawberry flavour . . . the best. Oh! There is a piece of tissue paper. Perfect, let’s switch to pen and paper (yes, I always carry a pen, we never know). Let’s think about our structure. Initially, we have no dots and no lines on the paper (D and L are both empty sets). So, we can add some dots to D. If we see there are too many, we can remove some dots. Likewise, we can draw some lines and add them to L or we can cross out some lines and remove them from L. So, those four operations (that is to say: addition of lines, subtraction of lines, addition of dots and subtraction of dots) can be performed on any graph G. If we want to adventure ourselves, it is also possible to draw one graph G_1 and another G_2 and connect them by adding a sufficient set of lines. We can also split a graph G by removing a sufficient set of lines from G such that we obtain two disconnected graphs G_1 and G_2.

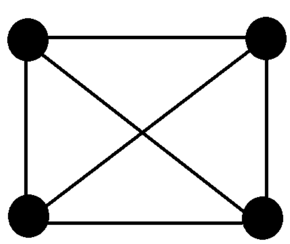
Now, not all graphs form a grid-like structure. In "dots and boxes", only horizontal and vertical lines are allowed between any two pairs of dots that are adjacent to one another. But a graph does not require such restrictions. In general, any dot can be connected to itself (by looping over itself) or connected to any other dots by any number of curved or straight lines. This yields a graph "zoo": different arrangements of dots and lines from which interesting structures (such as "trees" and "stars") emerge.



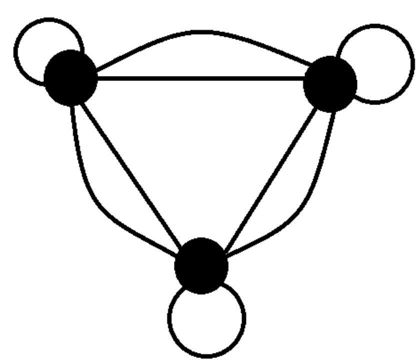
A Tree Graph



A Star Graph



A Complete Graph



A Non-Simple Graph

The best part is that graphs can get even fancier: we can add arrows to our lines to introduce an element of direction. We can assign a length to each line to get a sense of magnitude. We can even colour our dots and lines (who said that life must be monochrome?). This is looking too much fun now. It does not look like mathematics, it feels like a sandbox game in kindergarten. Surely that must be it, or are there more serious practical applications of the structure called a graph?

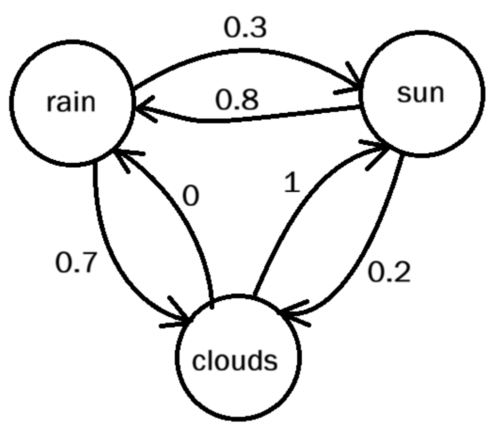
Sure to the pure mathematician, we could keep it at the sandbox level but the ubiquitous applications of the 'sandbox game' make it impossible to live in a world without graphs. In fact, everything depends on the meaning we attribute to those dots and lines. Let's discuss a few such meanings.

Global Positioning System (GPS)

Let's say we want to find the shortest path from Cannes to Paris. We could represent major stops between Cannes and Paris as dots**;** and the roads connecting the stops**,** as lines. Each road has a length and a traffic volume. We could assign a length to each line and colour each line to represent ranges of traffic volumes. For example, green lines could mean low traffic while red lines could mean high traffic. Having obtained our graph, we then apply algorithmic procedures such as that of Edsger Dijkstra to find the shortest path from Cannes to Paris.

Markov Chains

Markov Chains have states and transition probabilities from one state to the other. For example:



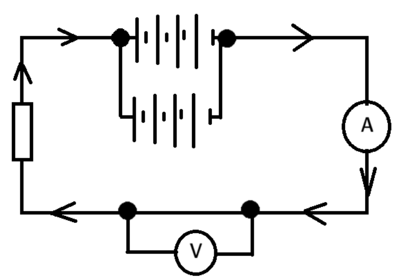
A Weather Markov Chain

Clearly, the above diagram is exactly a set of dots and lines.

Circulatory System

The heart pumps blood across our body. Each organ can be thought of as a dot connected by lines representing our arteries and veins. We also know that blood flows at different rates. A flow rate could be assigned to each line. The lines could further be coloured. Red lines could symbolize arteries while blue lines could represent veins.

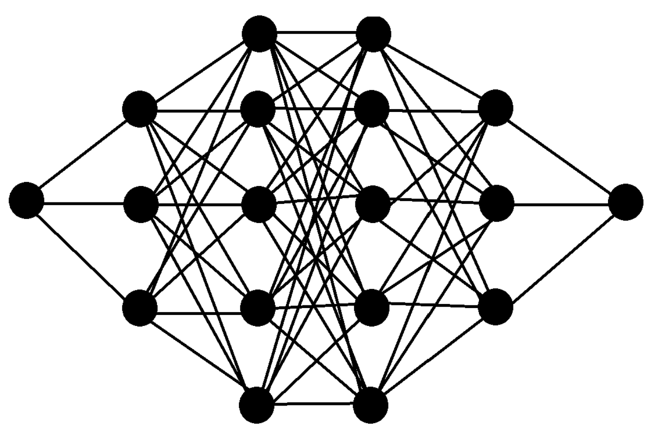
Electrical Circuit



An Electrical Circuit

Intuitively, wires symbolize lines while circuit components represent dots. In fact, it is from "dots and lines" that Gustav Kirchhoff, the great physicist, derived his laws of electricity.

Artificial Intelligence (AI)



A Neural Network

The dots and lines aspect is even more apparent in deep learning layers. Here, the dots represent mathematical functions while the lines represent feeding the output of one function as the input of another function.

Enough of applications! I think my breakfast has been digested (hopefully yours too!). We can now go for a swim.

But before we go, let's wrap up our conversation. Graphs, in fact, are pretty much like my dog, Rex. For what is a dog? An animal. Then what makes a dog a dog? A dog has hairs, eyes, a nose, … What does a dog do? A dog plays, eats, sleeps, … Are there different types of dogs? Yes, teckel, doberman, chihuahua, … Why do humans keep dogs? Well, dogs are very affectionate.

Likewise, what is a graph? A graph is a mathematical animal. What are the features of a graph? Dots, lines, colours, direction, … What does a graph do? Graphs can be added to one another, lose some lines, gain some dots, … Are there different types of graphs? Yes, trees, stars, cycles, … Why do humans study graphs? Well, finding optimal paths, predicting the future, AI, …

Overall, graphs provide a very powerful modelling tool and it is hard to find an entity in this world that cannot be modelled by graphs. Moreover, what is impressive is that it all stems from a simple idea of dots and lines that three-year old children draw all the time. However, notwithstanding the marvellousness of graphs, one would be wise not to become a graph-paranoid. After all, wouldn't performing depth-first-searches on real-life trees at the park, or trying to guess the Erdős number of your friend at a party, scare most people away?

That's it. Let's now have a nice rest of the day at the beach!

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