A Piece on Pi

Zachary Grant

April 3, 2026

1 What is Pi?

Pi (π) is the ratio between a circle's circumference and its diameter. Its earliest approximations date back almost 4000 years, when the ancient Babylonians estimated it to be 3.125. In the millennia that have followed, approximations have progressed to the point where, on the 11th of December 2025, 314 trillion digits of pi were calculated. In this essay, I will be exploring the mathematicians, and the mathematical discoveries, that have led us to this point.

2 How do you make a Polygon that big?

Archimedes of Syracuse was an ancient Greek mathematician who made waves - literally - in mathematics and beyond. When he was not building cutting edge war machines to overturn Roman ships, Archimedes was busy approximating pi, and creating a method of doing so that would be the basis of approximations for over a millennium. Archimedes approximated pi using 96-sided inscribed and circumscribed polygons, this same method was eventually used by Ludolph van Ceulen to make a 36 digit approximation, using a 2^{62}-sided polygon. Now, if you are anything like me, you might be thinking 'How do you make a Polygon that big?', well, let me explain.

If you search 'How did Archimedes approximate pi?' on YouTube or Google, you might come across a method similar to the one described below: Construct an n-sided polygon inscribed within a circle - radius r. Join each vertex to the centre of the circle, splitting the polygon into n isosceles triangles. Bisecting the vertex angle of one of these isosceles triangles then gives a right-angled triangle with angle $\frac{180}{n}^{\circ}$ at the centre of the circle. Then, some simple trigonometry will give you a perimeter of $2nr\sin(\frac{180}{n}^{\circ})$. Now, taking $r = \frac{1}{2}$ for a unit circle with perimeter π simplifies this down to $n\sin(\frac{180}{n}^{\circ})$. A construction of this can be seen below in Figure 1.

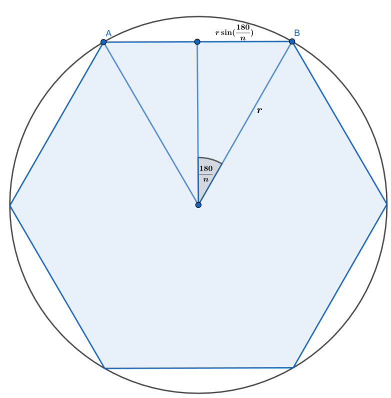

Figure 1: Inscribed Hexagon

A very similar process can be used to find the perimeter of an n-sided circumscribed polygon, and the formula for this comes out as $2nr\tan(\frac{180}{n}^{\circ})$ or $n\tan(\frac{180}{n}^{\circ})$ for a unit circle. Both of these equations seem to work, as n tends to ∞, both tend to π. You can even test this on your calculator, write either equation into your calculator and substitute in a large number - for example 2^{62} - and you will see that they both output π. Easy, right? What took these mathematicians so long? See, Archimedes' problem was that he did not have a modern day calculator - or the ability to calculate specific trigonometric values, meaning he could not have used this method.

So, now that we have busted a YouTube myth, let us see how Archimedes *actually* could've approximated pi.

We must first determine what tools and knowledge Archimedes had available to him. He had a strong understanding of geometry, mainly from Euclid's Elements. He also would have a basic understanding of trigonometry and the ratios between sides in some common triangles. Using this, Archimedes could have constructed something as seen below.

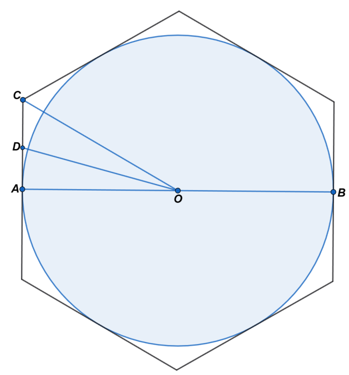

Figure 2: Circumscribed Hexagon

As $\angle AOC = 30°$, Archimedes could conclude that $\frac{OA}{AC} = \frac{\sqrt{3}}{1}$ and $\frac{OC}{AC} = \frac{2}{1}$. Archimedes could then have constructed an angle bisector of $\angle AOC$, forming triangle AOD. Using the angle bisector theorem from Book VI of Euclid's Elements, Archimedes could then have stated that $\frac{OC}{OA} = \frac{CD}{DA}$. Adding 1 to both sides gives $\frac{OC+OA}{OA} = \frac{CD+DA}{DA}$. As CD and DA are collinear $CD + DA \equiv AC$. Some rearranging of the above equation gives $\frac{OC+OA}{AC} = \frac{OA}{DA}$, substituting in the previously attained values for $\frac{OC}{AC}$ and $\frac{OA}{AC}$, we get $\frac{OA}{DA} = \sqrt{3} + 2$. Archimedes would then have used the Pythagorean theorem to find $\frac{OD}{AD}$ like so:

$$\frac{\sqrt{OA^2 + AD^2}}{AD} = \frac{OD}{AD}$$

It is important to note that Archimedes was unable to obtain exact values for irrational numbers such as $\sqrt{3}$, so he had to approximate them. Therefore, he would be able to get an approximation for $\frac{OD}{AD}$, and would be able to replace his initial values for $\frac{OA}{AC}$ and $\frac{OC}{AC}$ with his new-found values for $\frac{OA}{AD}$ and $\frac{OD}{AD}$, and repeat this process again. Each iteration of this process gives the ratios associated with a polygon with double the number of sides compared to the previous one. Archimedes repeated this four times, and was therefore able to estimate the perimeter of a 96-sided circumscribed polygon, giving him an upper bound for the value of pi, stating that $\pi < 3\frac{1}{7}$ Archimedes now had to obtain a lower bound for pi and would do so by estimating the perimeter of an inscribed polygon.

Below is a construction of an inscribed hexagon and dodecagon.

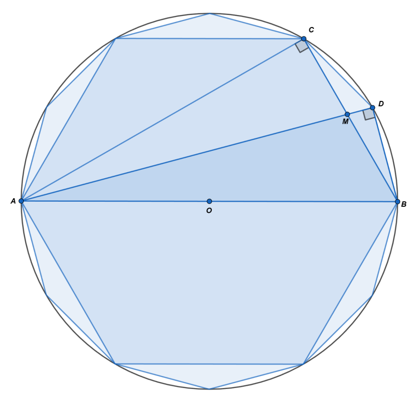

Figure 3: Inscribed Hexagon and Dodecahedron

As $\angle CAB = 30°$, it can be concluded that $\frac{AC}{BC} = \frac{\sqrt{3}}{1}$ and $\frac{AB}{BC} = \frac{2}{1}$. Triangles ACM,ADB, and BDM are all similar triangles. This means that the ratio between their sides are all equal.Therefore, we have $\frac{AD}{DB} = \frac{DB}{DM} = \frac{AC}{CM}$. Using the angle bisector theorem, we are also able to get $\frac{AC}{AB} = \frac{CM}{BM}$, which can be rearranged as $\frac{AC}{CM} = \frac{AB}{BM}$. Letting $\frac{AD}{BD} = \frac{AB}{BM} = \frac{AC}{CM} = k$ where k is some constant, we can find that $AC = kCM$ and $AB = kBM$, adding these gives $AB + AC = k(CM + BM)$ and therefore $k = \frac{AD}{DB} = \frac{AB+AC}{CM+BM}$. As BM and CM are collinear, this can be simplified down to $\frac{AB+AC}{BC} = \frac{AD}{BD}$. Using the ratios for $\frac{AB}{BC}$ and $\frac{AC}{BC}$ it would have been possible for Archimedes to get an estimate for $\frac{AD}{BD}$, he could then have used the Pythagorean theorem like so

$$\frac{\sqrt{AD^2 + DB^2}}{DB} = \frac{AB}{BD}$$

Archimedes was now able to use the ratios of $\frac{AD}{BD}$ and $\frac{AB}{BD}$ as his new starting point, and repeat the process. Archimedes repeated this process four times, eventually finding the ratios associated with a 96-sided inscribed polygon. Archimedes could therefore get a lower bound for pi, and stated that $3\frac{10}{71} < \pi$. Combining this with his upper bound, Archimedes stated that "The ratio of the circumference of any circle to its diameter is greater than $3\frac{10}{71}$ but less than $3\frac{1}{7}$" (in Ancient Greek, obviously). This estimate gave the first three digits of pi for certain, 3.14. This is an approximation within 0.05% of the true value of pi, a ridiculously impressive feat, given that it was published over 2200 years ago.

This method was used over the next 1800 years by mathematicians with a lot of dedication and even more free time. Ludolph van Ceulen used a polygon with 2^{62} sides to produce a 35 digit approximation of pi, a process which took him 25 years. This was an achievement he was so proud of that he got his upper and lower bounds of pi inscribed on his tombstone. Whilst this method involves some ingenious geometry, it is painstakingly slow. Not every mathematician had the dedication of Ludolph van Ceulen, and this led to some mathematicians looking for a faster way and, as great mathematicians do, they found one.

3 To infinity and beyond

The next major breakthrough in pi approximations came in Kerala in the 14th century. Indian mathematician Madhava of Sangamagrama found the Maclaurin series for arctangent, a discovery so ahead of it's time that it was not rediscovered for 200 years, when James Gregory and Gottfried Liebniz found it.

Madhava, Leibniz and Gregory all used methods involving complex geometry and calculus, so we shall derive the Madhava-Leibniz series in a simpler way using modern methods. Using the fact that $\tan(\arctan(x)) = x$, we can let $y = \arctan(x)$ and therefore $\tan(y) = x$. Differentiating both sides with respect to x gives $\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x)$, which simplifies to $\sec^2(y)\frac{dy}{dx} = 1$ solving for $\frac{dy}{dx}$ and using the identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, you can then find that $\frac{dy}{dx} = \frac{1}{1+\tan^2(y)}$, as $\tan(y) = x$, this is equivalent to $\frac{dy}{dx} = \frac{1}{1+x^2}$. Expanding $\frac{1}{1+x^2}$ using the binomial expansion gives $1 - x^2 + x^4 - x^6 + ...$ Therefore, $\frac{d}{dx}[\arctan(x)] = 1 - x^2 + x^4 - x^6 + ...$, integrating both sides gives

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

This is the Gregory series expansion for arctangent. Using the fact that $\arctan(1) = \frac{\pi}{4}$, it is possible to get an approximation for pi using this series:

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

This was the first infinite series that was used to approximate pi, although it converged very slowly, making it quite inefficient. By making some tweaks and using more efficient series Madhava made an 11 digit approximation in approximately 1400, a record at the time. This infinite series, along with many others, were used to approximate pi for hundreds of years, and similar methods are still used by computers to create modern approximations. In 1706 John

Machin used the Gregory series for arctangent to calculate the first 100 digits of pi, and in the same year William Jones was the first person to assign the Greek letter π to the ratio of the circumference of a circle to its diameter.

4 Man and Machine

In 1950, the first computer approximation of pi was calculated to 2038 digits, and in 2025 the record was set at 314 trillion digits. While modern computers are able to perform calculations at speeds that are unimaginable to humans, they still would not be able to do it without the spark of human ingenuity. In 1910, perennial overachiever Srinivasa Ramanujan found many rapidly converging series for pi, such as

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

These series were later modified by the Chudnovsky brothers and were used to approximate pi to over 1 trillion digits. The fact that a mathematician, with almost no formal mathematical training, created a series that was the basis of pi approximations for decades to follow is an example of not only the genius of Ramanujan, but the importance of humans in mathematics, a topic that feels especially poignant with the recent developments in artificial intelligence.

5 Was this all rational?

NASA use a value of pi to 15 decimal places to perform their calculations, a value that was surpassed 600 years ago by Jamshid al-Kashi. If one of the leading space exploration agencies in the world only need 15 decimal places, what have the last 600 years been for? Did Ludolph van Ceulen need to dedicate 25 years of his life to improve an approximation that was already good enough? Whilst the rational answer is no; mathematicians, just like pi itself, are not rational. The story of pi approximations is, at its core, a story of mathematicians' competitiveness, ingenuity, and thirst for greater understanding, traits that underpin all mathematical discoveries.

6 Bibliography

Archimedes (n.d.). ARCHIMEDES, MEASUREMENT OF A CIRCLE 1. [online] Available at: `https://media.bloomsbury.com/rep/files/primary-source-13-archimedes-measurem pdf`.

Ciaran McEvoy (2021). How did Archimedes ACTUALLY calculate pi? Pi day 2021. [online] YouTube. Available at: `https://www.youtube.com/watch?v=B9I22ab991o` [Accessed 4 Jun. 2024].

Joanna (2018). Celebrating Pi Day - Ludolph van Ceulen, another forgotten hero of Pi. [online] Maths Careers. Available at: https://www.mathscareers.org.uk/celebrating-pi-day-ludolph-van-ceulen/.

NASA/JPL Edu (2022). How Many Decimals of Pi Do We Really Need? [online] NASA Jet Propulsion Laboratory (JPL). Available at: https://www.jpl.nasa.gov/edu/news/how-many-decimals-of-pi-do-we-really-need/.

Roberts, D. (2026). Angle Bisector Theorem - MathBitsNotebook(Geo). [online] Mathbitsnotebook.com. Available at: https://mathbitsnotebook.com/Geometry/Similarity/SMAngle.html [Accessed 6 Mar. 2026].

Roy, R. (n.d.). -Sources in the Development of Mathematics: Infinite Series and Products from the Fifteenth to the Twenty-first Century Ranjan Roy Excerpt More information 1 Power Series in Fifteenth-Century Kerala. [online] Available at: https://assets.cambridge.org/97805211/14707/excerpt/9780521114707_excerpt.pdf.

Wikipedia Contributors (2019). Archimedes. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Archimedes.

Wikipedia Contributors (2024). Arctangent series. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Arctangent_series.

Wikipedia Contributors (2026a). Approximations of pi. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Approximations_of_pi#16th_to_19th_centuries.

Wikipedia Contributors (2026b). Chronology of computation of pi. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Chronology_of_computation_of_pi.Wikipedia.

(2022). Madhava series. [online] Available at: https://en.wikipedia.org/wiki/Madhava_series.