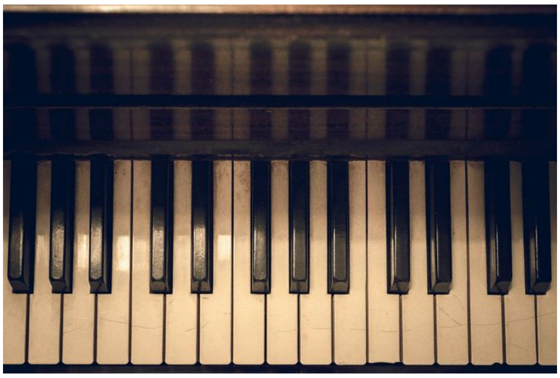
The Magic of Fibonacci Numbers
And the Golden Ratio in Music



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1.1 Understanding Fibonacci Numbers

[*What is the Fibonacci Sequence?*]

The fibonacci sequence is a series of numbers in which each number (Fibonacci number) is the sum of the two preceding numbers.

0 , 1 , 1 , 2 , 3 , 5 , 8 , 13 , 21 , 34 , 55 , 89 …

Each number, starting with the third, follows the formula. For example;

1 + 1 = 2 / 2 + 1 = 3 / 3 + 2 = 5

[*Calculating the Fibonacci Sequence]*

In this formula, every number in this sequence is considered a "term", represented by F_n. The "n" denotes the number's position in the sequence, starting with zero. For example;

8th Term (13) = $\mathbf{F_7}$ / *9th Term (21)* = $\mathbf{F_8}$

The fibonacci sequence then can be defined by three equations:

1) $\mathbf{F_0 = 0}$ (applies only to the first term)
2) $\mathbf{F_1 = 1}$ (applies only to the second term)
3) $\mathbf{F_n = F_{n-1} + F_{n-2}}$ (applies to all others)

For example;

$\mathbf{F_{10} = F_{10-1} + F_{10-2} = F_9 + F_8 = 34 + 55 = 89}$

The problem with this formula is that you need to know the previous 2 numbers, to get the third.

1.2 Binet's Formula

To *fix* this problem, a french mathematician named Jacques Phillipe Marie Binet created an explicit, closed form formula to find the nth term of the Fibonacci sequence.

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Proof:

We know that the equation for the sequence is $\mathbf{F_n = F_{n-1} + F_{n-2}}$. We assume that $F_n = r_n$, making a recurrence (a sequence that gives you a connection between two consecutive terms):

$$\mathbf{r_n = r_{n-1} + r_{n-2}}.$$

Divide both sides by $\mathbf{r_{n-2}}$ and rearrange:

$$\mathbf{r^2 - r - 1 = 0}$$

This forms the quadratic equation:

$$x^2 - x - 1 = 0$$

This quadratic has the roots:

$$\frac{1 \pm \sqrt{5}}{2}$$

$\sigma = \frac{1+\sqrt{5}}{2}$ (the golden ratio), $\tau = \frac{1-\sqrt{5}}{2}$

The quadratic can also be written as:

$$x^2 = x + 1$$

Write expressions for x^n

$$\begin{aligned}
x &= x \\
x^2 &= x + 1 \\
x^3 &= x \cdot x^2 \\
&= x \cdot (x + 1) \\
&= x^2 + x \\
&= (x + 1) + x \\
&= 2x + 1 \\
x^4 &= x \cdot x^3 \\
&= x \cdot (2x + 1) \\
&= 2x^2 + x \\
&= 2(x + 1) + x \\
&= 3x + 2 \\
x^5 &= 5x + 3 \\
x^6 &= 8x + 5
\end{aligned}$$

Therefore, a pattern can be noted.

[using numbers from the Fibonacci e.g., $F_1 = 1$, $F_2 = 1$, $F_3 = 2$…]

$x^1 = 1x + 0 = F_1x + F_0$

$x^2 = 1x + 1 = F_2x + F_1$

$x^3 = 2x + 1 = F_3x + F_2$

$x^4 = 3x + 2 = F_4x + F_3$

$x^5 = 5x + 3 = F_5x + F_4$

These sets of equations give us:

$$\mathbf{x^n = F_nx + F_{n-1}}$$

Since the roots of the quadratic are:

$\sigma = \frac{1+\sqrt{5}}{2}$ (the golden ratio), $\tau = \frac{1-\sqrt{5}}{2}$

They must both satisfy the equation:

$$\sigma^n = F_n\sigma + F_{n-1}$$

$$\tau^n = F_n\tau + F_{n-1}.$$

To find F_n, subtract the second equation from the first.

$$\sigma^n - \tau^n = F_n(\sigma - \tau) + F_{n-1} - F_{n-1}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n = F_n\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

Solve this equation, and it yields the general formula for the nth Fibonacci Number:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Does this formula work?

$$F_3 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3}{\sqrt{5}}$$

$= \mathbf{2}$

$$F_{30} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{30} - \left(\frac{1-\sqrt{5}}{2}\right)^{30}}{\sqrt{5}}$$

$=$ **832040**

While mathematically correct, it can create rounding errors when using large numbers.

1.3 Number Patterns in the Fibonacci

The Fibonacci sequence does not just consist of the addition of numbers, but rather, a series of patterns.

Let's start with squaring the entire sequence:

1	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**…**
1	1	4	9	25	64	169	441	1156	**…**

Notice how:

1	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**…**
1	1	4	9	25	64	169	441	1156	**…**

If you add the bottom squared numbers, you get a number of the **original series.**

1 + 1 = **2** / 1 + 4 = **5** / 4 + 9 = **13** / 9 + 25 = **34**

However, what happens when you add the *first few* squared numbers together?

1 + 1 + 4 = 6 = 2 x 3

1 + 1 + 4 + 9 = 15 = 3 x 5

1 + 1 + 4 + 9 + 25 = 40 = 5 x 8

1 + 1 + 4 + 9 + 25 + 64 =104 = 8 x 13

At first, these numbers may not seem like they are part of the sequence, but in reality, they are composed of Fibonacci Numbers.

Now create more pairs of Fibonacci numbers, and divide the bigger numbers by the smaller numbers:

8 x 13 — 13/8 = 1.625

13 x 21 — 21/13 = 1.615…

21 x 34 —- 34/21 = 1.619…

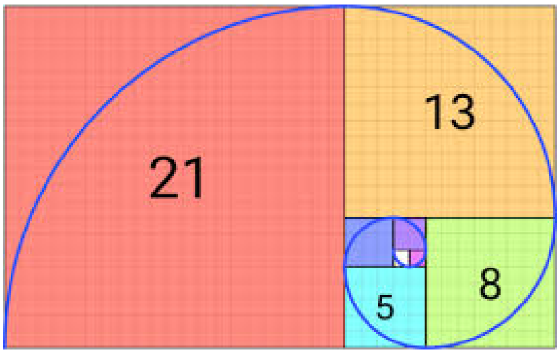
34 x 55 —- 55/34 = 1.6176…

55 x 89 —- 89/55 = 1.61818…

These numbers get closer and closer to **1.1618033,** which is known as the *Golden Ratio.*

1.4 The Golden Ratio

The golden ratio is a relationship between two numbers that are next to each other in the Fibonacci Sequence. This golden ratio, at approximately *1.1618033*, creates a golden spiral that is considered a benchmark for aesthetically pleasing images.



This image shows the Fibonacci Spiral, which is an approximation of the Golden Spiral. It uses quarter arcs inscribed within squares of the Fibonacci numbers.

The golden spiral looks similar, but is more mathematically precise, and is directly based on the golden ratio: $\phi \approx 1.618$. It is a specific type of logarithmic spiral, that expands outwards by the factor of ϕ, for every 90 degrees. This means the as the spiral winds around its center, the distance from the center increases by a factor of ϕ after each quarter turn.

The general equation of a logarithmic spiral is:

$$\mathbf{r = ae^{b\theta}}$$

r = distance from the origin,

a = a positive real constant that sets the initial size of the spiral

e = Euler's number

b = determines the rate of exponential growth of the spiral

θ = angle in radians

For the golden spiral, we want it to grow by a factor of ϕ every 90 degrees, which is $\pi/2$ radians.

To find the value of b (the rate of exponential growth for the spiral): here are the steps

Step 1:

When θ increases by $\pi/2$, we want r (the radius to grow by ϕ.

$$r(\theta + \pi/2) = \phi * r(\theta)$$

Step 2:

Plug this into the spiral formula $r = ae^{b\theta}$

At angle θ, the radius is:

$$r(\theta) = ae^{b\theta}$$

At angle $\theta + \pi/2$, the radius is:

$$r(\theta + \pi/2) = ae^{b(\theta + \pi 2)} = ae^{b\theta} * e^{b\pi/2}$$

This also refers to an exponential property:

$$e^{x+y} = e^x * e^y$$

Step 3:

Now divide $r(\theta + \pi/2)$ by $r(\theta)$: (notice how $ae^{b\theta}$ cancels out)

$$\frac{r(\theta+\pi/2)}{r(\theta)} = e^{b\pi/2}$$

Step 4:

Solve for b with the golden ratio

$$e^{b\pi/2} = \phi$$

We do this by taking the natural logarithm (ln) on both sides – $\ln(e^x) = x$, which helps us get rid of the exponential

$$\ln(e^{b\pi/2}) = \ln(\phi)$$

$$b\pi/2 = \ln(\phi)$$

Now solve for b:

$$b = \frac{\ln(\phi)}{\pi/2}$$

Use calculator values:

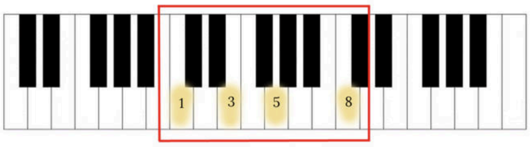
$$b = \frac{0.4812}{1.5708} \approx 0.3063$$

–The golden spiral, which is based on the Golden Ratio, is used in various areas like art, architecture, design and music to create visually pleasing "compositions, arrangements and melodies. It can also be found in nature, in the arrangements of seashells, pinecones, sunflowers and other species. What's most interesting is how the fibonacci and the golden ratio both have heavy influences in music.

1.5 Fibonacci Sequence and Golden Ratio in Music

When listening to captivating film scores or illuminating sonatas, many wonder whether there is a mathematical way to write music.

While there is no evidence of the legendary John Williams using these mathematical methods, there is evidence that suggests that composers like Mozart and Debussy used the fibonacci sequence to write their sonatas.



For example:

a) An octave on the piano consists of *13* notes: *8* white keys and *5* black keys

b) A scale consists of *8* notes, of which the *3rd* and *5th* notes make up a basic chord

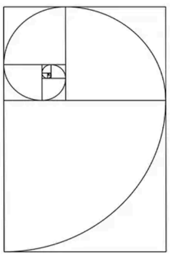
c) In a scale, the dominant note is the *5th* note, which is also the *8th* note of all *13* notes that make up the octave

d) 8 divided by 13 equals 0.61538… which is the approximate golden ratio.

Notice: these are all numbers which appear in the Fibonacci Sequence.

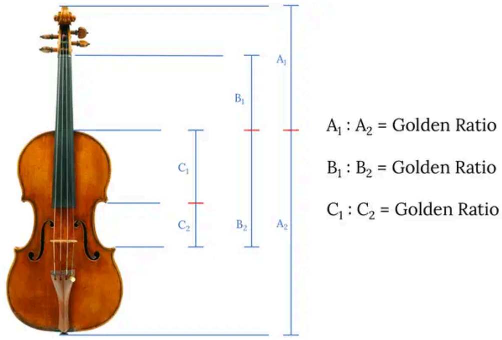
Golden Spiral:
The fibonacci sequence can be transformed into the golden spiral, which is also used in music.

For example, the **bass clef** uses this spiral:



It is also used when creating instruments. The Master of violin making, Antonio Stradivari has made some of the greatest and most beautiful violins in all of existence.

However, there is a reason a Stradivarius violin would cost a few million pounds – and its value comes from how it was created – using the fibonacci sequence and its Golden Ratio.

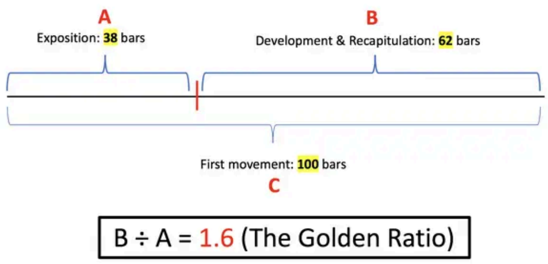


How did Mozart and Debussy use this?
Mozart used this ratio in his piano sonatas quite frequently. A traditional sonata has two parts:

a) *Exposition* – where the music is introduced
b) *Development and Recapitulation* – where the theme is developed and repeated

Mozart arranged his piano sonatas so that the number of bars in the *development and recapitulation* divided by the number of bars in the *exposition* would equal about 1.618, which is the Golden Ratio.

For example: Mozart's Piano Sonata No.1 in C Major



In the diagram, C is the sonata's first movement as a whole, B is the *development and recapitulation,* and A is the *exposition*.

The *exposition* consists of 38 bars and the *development and recapitulation* consists of 62.

62 / 38 = 1.632 (approximately the golden ratio)

Some claim that Beethoven, Bartók, Debussy, Schubert, Bach and Satie also used this technique to write their sonatas, but no one is quite sure why it works so well.

Golden Ratio as a Climax:
Often so, the use of the Golden Ratio isn't just limited to classical music, but rather has been used in genres like Pop and Rap. For musicians, it's relevant in terms of when writing a piece of music, where to place the climax. Obviously, they wouldn't place it *right* in the centre of the piece, but a little "off-center", gravitating towards the Golden Ratio.

Musicians call this the **"Phi Moment":**

*[LENGTH OF THE SONG] * 0.618*
= the exact moment
(many musicians put their final chorus at this moment)

For example: (Different Genres)

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1) **POP:**

That's so True - Gracie Abrams

*165s * 0.618 = 101.97*
101.97/60 = 1.69 (2:09)
At 2:09, she reaches her final chorus/climax, which finishes the rest of the song.

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2) **R&B**

Love Me Not - Ravyn Lenae

*213s * 0.618 = 131.634*
131.634 / 60 = 2.19
At 2.19 she reaches her bridge (which is the climax)

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3) **Rock**

Mr Brightside - The Killers

*223s * 0.618 = 137.814*
137.814 / 60 = 2.29
This one is the most exact, at precisely 2:29, they begin their final chorus, finishing the song.

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4) **Rap**

SICKO MODE - Travis Scott

*312s * 0.618 = 192.816*
192.816 / 60 = 3.21
At exactly 3:21, he begins his bridge, which is completely different from the rest of the song.

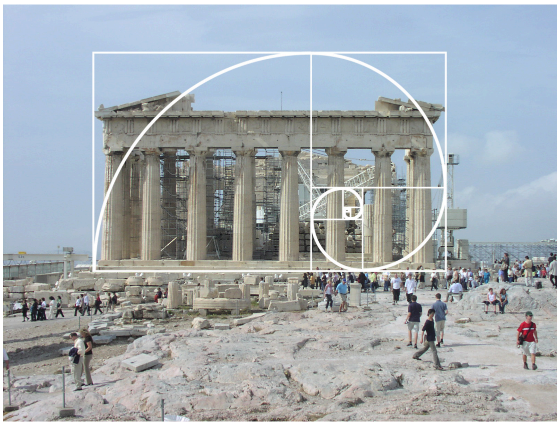
1.6 Conclusion

While present in many instances, many still believe that the Golden Ratio is a myth, a value that truly can never be reached nor proven. This is because:

The Ratio Will Always Be Off

If you have two objects, and if after you do the math, get 1.6180, it's usually accepted that those two numbers are part of the Golden Ratio. Except, that isn't the Golden Ratio. The ratio is 1.61803…, and the decimal points go forever.

Technically then, it is impossible for anything in the real world to fall under the golden ratio. It's almost like Pi. Many say that there are no perfect circles, as they're always going to be a little off.



For example, strictly speaking, you can fix anything into a golden spiral. In this image, the photo isn't *really* fitting the golden spiral, but can be oriented to do so.

If it really is that flimsy, then why does it persist?

It's simple. Humans are genetically programmed to seek patterns and seek meaning, it's too frivolous to find something "aesthetically pleasing", so we back it up with our limited grasp of math. Most people don't even *understand* the golden ratio, so they can't correct themselves.

However, artists like Leonardo Da Vinci actively utilised it in their art, as well as in music.

Does something need to be the precise number to be considered part of the ratio?

1.7 Bibliography

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