What If $ax + b$? A Guide to the Collatz Multiverse

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Dedicated to my dad, who taught me to love mathematics. Rest in peace.

Introduction: The Original Collatz Universe

Ever since I first heard about it, I have been fascinated by the Collatz Conjecture. It is such a simple problem, and yet it remains unsolved.

Here's the conjecture:

If the number is odd, multiply it by 3 and add 1.

If the number is even, divide it by 2.

Such a simple rule leads to a surprising conjecture: that every number eventually reaches 1.

Let's start with an example: 5.

5 is odd. Multiply by 3 and add 1 to get 16. Since 16 is even, we divide it by 2.

5 → 16 → 8 → 4 → 2 → 1.

At this point, we reach the loop 1 → 4 → 2 → 1.

Interesting, right? Try 7 for yourself. Why 7? It doesn't really matter. If you want a longer journey, try 27.

This naturally led me to a question. What if this conjecture was just one of many? What would happen if we change the rules? Why multiply by 3 and add 1? Why not other numbers? Why not negative numbers?

That is exactly what this essay sets out to explore. So, strap in, we shall now explore the *Collatz Multiverse*.

Exploring New Universes

The Collatz Conjecture, or $3x + 1$, is just one part of a bigger family tree. If Collatz has been explored so extensively, then think of the other universes! What would the behaviour be there? Would it shrink to 1? Would there be interesting characteristics?

To answer these questions, let us consider the general form $ax + b$. Here a and b are odd integers. In this essay, x represents an integer. For now, we will continue to divide even numbers by 2 (though other numbers are worth exploring).

Now you might be wondering: why should a and b be *odd* integers specifically? The reason for this choice will become clearer later on in the essay, as it plays a key role in how the sequence behaves.

To begin exploring these universes, I made a small change. Instead of $3x + 1$, I tried the rule $3x - 1$.

Let's test out 7.

7 → 20 → 10 → 5 → 14 → 7

What a surprise: it loops back. This is interesting, as a small change to the rule has led to a completely different outcome. But is this example an exception?

Let’s try 11.

11 → 32 → 16 → 8 → 4 → 2 → 1 → 2

Here we have found another result. It does go down to 1 but the loop is 1 → 2 instead of the 4 → 2 → 1 cycle seen earlier.

After further investigation, we find many loops, not all of which lead to 1. But we have found another type of universe. Let us call these *Loop Universes*. Universes like the original Collatz rule appear to shrink toward 1. Although the Collatz rule has not been proven, for simplicity I shall be calling them *Shrinking Universes*.

Now that there are already two types, this naturally raises the question: are there more?

To explore this further, and to answer the question of whether more types exist, I experimented with another variation: $5x + 1$.

By trying out 9, we see this path:

9 → 46 → 23 → 116 → 58 → 29 → 146 → 73 → 366 → 183 → …

As we can see, it continues to grow, and may even tend towards infinity. But not all starting numbers behave this way, and some even end up in loops. This suggests the existence of a new type of universe: *Growing Universes*. In these universes, sequences can grow very large without immediately shrinking or looping.

Can We Predict a Universe?

At this point I began to wonder if it was possible to predict the behaviour of the universes. One way to think about this is to consider the average effect of each step.

When a number is even, it is divided by 2, which is equivalent to multiplying by ½. When the number is odd, it is transformed by the rule $ax + b$, which roughly behaves like multiplying by a factor of a, especially for larger numbers.

However, after applying the $ax + b$ rule, the result is often divisible by 2 multiple times. This means that one odd step ($ax + b$) is often followed by multiple even steps (/2). For some numbers this may happen only once; for others, it may happen several times.

This means that it is difficult to predict the behaviour of the universes. However, we can use the average effect to estimate the behaviour. If, on average, a number is multiplied by a factor less than 1, the sequence tends to shrink. And if it is multiplied by a factor greater than 1, the sequence tends to grow.

Something to keep in mind is that this is only a rough guess and is not always true. For example, in the $5x + 1$ universe, the odd step increases the number significantly but this is often followed by multiple divisions of 2. As a result, the overall effect is not a fixed factor, and sequences can sometimes grow and other times shrink, often leading to loops.

Why Restrict to Odd Coefficients?

At this point, it is worth returning to an earlier question: why did we restrict a and b to just odd integers?

It helps if we look at the parity of $ax + b$ when x is odd, since even numbers are halved repeatedly until they reach an odd number.

There are three important cases:

<u>Case 1</u>: Both a and b are odd

In this case, ax will be odd since odd × odd = odd. Adding another odd number gives an even result. This means that every odd step is guaranteed to be followed by at least one division by 2. This preserves the characteristic Collatz rhythm of growth followed by reduction.

<u>Case 2</u>: One of a and b is odd while the other is even

Here, the result is always odd.

If a is odd and b is even, then ax is odd and $ax + b$ remains odd.

If a is even and b is odd, then ax is even and $ax + b$ is again odd.

So, in either case, an odd input produces an odd output. This means that the sequence never switches to the division step. For example, under the rule $2x + 1$, starting from 5 gives 5 → 11 → 23 → 47 →..., which stays odd throughout. The rise-and-fall of the Collatz process disappears, so these universes are usually less interesting.

<u>Case 3</u>: Both *a* and b are even.

In this case, the resulting universe is often just a "*copy*" of a simpler one. This is because when x is odd, $ax + b$ = $2(Ax + B)$ for some integers A and B, so the result is automatically even. This means that after one division by 2, the sequence simply follows the rule $Ax + B$.

This means that this case can contain copies of both previous cases.

For example, the universe $6x + 2$, which is 2($3x + 1$), behaves exactly like $3x + 1$, except that every odd step includes one extra halving step.

Similarly, the universe $4x + 2$, which is 2($2x + 1$), behaves like $2x + 1$, again with one extra halving step.

In this sense, even coefficients often do not create genuinely new universes, but only redundant versions of ones we already know.

By restricting a and b to odd integers, we reduce repetition and focus on the most interesting cases, where growth and division interact in complex and less predictable ways.

In other words, odd coefficients preserve the delicate balance between growth and reduction that gives Collatz its mystery.

Negative Universes

One final direction worth mentioning is the existence of *negative universes*, where either a or b is negative. We already saw this in the $3x - 1$ universe, which produced loops very different from the original Collatz rule. Allowing more negative values opens up even stranger worlds, where numbers could bounce between positive and negative values, fall into unexpected cycles, or escape in entirely different ways. Although I have only briefly explored these universes, they suggest that the Collatz Multiverse may be far richer than it first appears. The task of uncovering their quirks is perhaps best left as an exercise to the reader.

Conclusion: A Mathematical Multiverse

The Collatz Conjecture begins with a simple rule, yet a few small modifications reveal a far richer landscape. Instead of a single problem, we uncover a whole collection of mathematical universes: some shrink, some loop, some grow, and some mirror others as redundant copies. What is most striking is how sensitive these universes are to tiny changes. Changing +1 to −1, replacing 3 with 5, or altering the parity of the coefficients can completely transform the behaviour of the sequence. The same simple mechanism can give rise to order, chaos, repetition, or even explosive growth.

This is what makes the Collatz Multiverse so fascinating. A rule simple enough to explain in one sentence unfolds into a vast family of strange and unpredictable worlds, each with its own rhythm and personality — and apparently, enough mystery to inspire an essay over 1500 words long. Perhaps that is the true beauty of mathematics: even the simplest ideas can contain entire universes waiting to be explored.