<u>Algebraic manipulations of floor function</u>

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"I learned algebra... knowing the whole idea was to find out what was and it didn't make any difference how you did it."
– Richard Feynman

Introduction

The aim for this paper is to explore, sometimes ill-taught, manipulation techniques used for transforming discontinuous functions, especially the floor/Greatest Integer Function (G.I.F.) .I refer to them as ill-taught because of the lack of coverage given to these functions in modern school syllabi and fixating on continuous functions and their algebra.

In the following manner (indexing) we are going to take a look at this wonderful world of "floor algebra".

I. Motivation
II. Probable approaches and issues with them
III. Solution of the problem
IV. Main takeaways
V. Conclusion

Motivation

When I was wrestling with floor functions, especially of the form where the variable is divided by constants, I came across an identity which related the floor of n/5 to some nested floors of n/10, namely,

$$\left\lfloor \frac{n}{5} \right\rfloor = 2\left\lfloor \frac{n}{10} \right\rfloor + \left\lfloor 0.1\left(n - 10\left\lfloor \frac{n}{10} \right\rfloor\right) + 0.5 \right\rfloor$$

Deriving this equation is trivial, but for the sake of completeness here is a logical way to derive this equation.

The first preposition for deriving this identity is that for many cases (last digit of n<5) the number of multiples of are two times the number of multiples of 10

Secondly, if the last digit is greater than or equal to 5 then an extra number becomes a multiple of 5. So if we were allowed to use piece-wise definition, we could use the definition

$f(n) = 2\left\lfloor \frac{n}{10} \right\rfloor$; If last digit of n is less than 5

$f(n) = 2\left\lfloor\frac{n}{10}\right\rfloor + 1$; If last digit of n is greater than or equal to 5

To convert these two functions into a single one, we use the fact that the straight line represented by $y = 0.1x + 0.5$ attains values less than zero for all single digit integers less than 5 and attains the value of one for all single digit integers greater than 5.

Therefore the final equation that expresses our function is,

$$f(n) = 2\left\lfloor\frac{n}{10}\right\rfloor + \left\lfloor 0.1(n - 10\left\lfloor\frac{n}{10}\right\rfloor) + 0.5\right\rfloor$$

Here ($n - 10\left\lfloor\frac{n}{10}\right\rfloor$) yields the last digit (units place) of the number n.

As the final step, we have to observe that by definition of f(n), it must yield the same result as $\left\lfloor\frac{n}{5}\right\rfloor$

Therefore,

$$\left\lfloor\frac{n}{5}\right\rfloor = 2\left\lfloor\frac{n}{10}\right\rfloor + \left\lfloor 0.1\left(n - 10\left\lfloor\frac{n}{10}\right\rfloor\right) + 0.5\right\rfloor$$

As you can see that with some hand waving we have successfully "derived" the above identity

Now our curiosity might lead us to question whether it is possible to use a set of rules to transform the right hand side to left hand side. To do so here are some of my initial thoughts that helped me understand the structure of the problem.

Approaches

1. **Decomposing n into fractional and integer part**
   For this approach we might think of dividing n into fractional part and integral part, this is a really common trick taught in many course works which students often "memorize". This approach goes like:
   Find the variable, substitute the sum of fractional and integral part of the variable and then simplify.
   Easy? Let's check.
   $n = \lfloor n\rfloor + \{n\}$
   It might come across our mind that the identity wasn't checked for non integral values especially the function we are using to extract unit digit of n, it produces gibberish when non integral values are involved.
   But still if we decide to carry along hoping those fractional parts might cancel out somewhere, it results in the following equation
   $$\left\lfloor\frac{\lfloor n\rfloor + \{n\}}{5}\right\rfloor = 2\left\lfloor\frac{\lfloor n\rfloor + \{n\}}{10}\right\rfloor + \left\lfloor 0.1\left(\lfloor n\rfloor + \{n\} - 10\left\lfloor\frac{\lfloor n\rfloor + \{n\}}{10}\right\rfloor\right) + 0.5\right\rfloor$$

We can easily remove the fractional part inside floor function, right? Well actually, the result isn't wrong intrinsically, but the it only works because the denominator is a integer (maybe this is an exercise for the reader to check why this only works when denominator is an integer). Now the equation becomes:

$$\left\lfloor \frac{\lfloor n \rfloor}{5} \right\rfloor = 2\left\lfloor \frac{\lfloor n \rfloor}{10} \right\rfloor + \left\lfloor 0.1\left(\lfloor n \rfloor + \{n\} - 10\left\lfloor \frac{\lfloor n \rfloor}{10} \right\rfloor\right) + 0.5 \right\rfloor$$

Observing this equation, it can't be further simplified. So let's keep this in the back of our mind and think of another way we could approach this problem.

2. **Reducing nested floor functions**

Let us suppose that we have $k = \left\lfloor \frac{\lfloor \frac{x}{m} \rfloor}{n} \right\rfloor$,

Now to reduce this function we might use the formal definition of floor function which is defined using inequalities.

Floor(x) is defined to be some n belonging to the set of integers such that $n \leq x < n + 1$.

$$k \leq \frac{\left\lfloor \frac{x}{m} \right\rfloor}{n} < k + 1$$

$$nk \leq \left\lfloor \frac{x}{m} \right\rfloor < n(k + 1)$$

Now splitting this inequality and again applying the definition of floor function results in,

$$k \leq \frac{x}{mn} < k + 1$$

This result proves that k must be the value of the floor function of x/(nm). From the initial proposition we can say that the two floor functions must be equal.

Hence,

$$\left\lfloor \frac{\left\lfloor \frac{x}{m} \right\rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{mn} \right\rfloor$$

Did you observed that this could prove the preposition which was given at the end of previous subtopic?

Now does it work for our problem?

Let's check.

$$\left\lfloor \frac{n}{5} \right\rfloor = \boxed{2\left\lfloor \frac{n}{10} \right\rfloor} + \left\lfloor 0.1\left(n - \boxed{10\left\lfloor \frac{n}{10} \right\rfloor}\right) + 0.5 \right\rfloor$$

The boxed parts can be reduced using the recently derived property.

$$\left\lfloor \frac{n}{5} \right\rfloor = \left\lfloor \frac{n}{5} \right\rfloor + \lfloor 0.1(n - \lfloor n \rfloor) + 0.5 \rfloor$$

But this must imply that the error term(for that error by 1) must be zero. It must mean that we have messed up while applying it.

The error again arises due to the denominators not being integers. With this mistake, we will, from now on, get alerted as soon as we get a non integral value in the denominator.

Solution

As in the first approach we are going to substitute n to be

$$n = 10q + r$$

r is between zero and 9

$$\left\lfloor \frac{10q + r}{5} \right\rfloor = 2\left\lfloor \frac{10q + r}{10} \right\rfloor + \left\lfloor 0.1\left(10q + r - 10\left\lfloor \frac{10q + r}{10} \right\rfloor\right) + 0.5 \right\rfloor$$

This equation simplifies to,

$$\left\lfloor 2q + \frac{r}{5} \right\rfloor = 2\left\lfloor q + \frac{r}{10} \right\rfloor + \left\lfloor 0.1\left(n - 10\left\lfloor q + \frac{r}{10} \right\rfloor\right) + 0.5 \right\rfloor$$

Now q and 2q are integers hence they can me taken out of the floor function.

$$2q + \left\lfloor \frac{r}{5} \right\rfloor = 2q + 2\left\lfloor \frac{r}{10} \right\rfloor + \left\lfloor 0.1\left(n - 10q - 10\left\lfloor \frac{r}{10} \right\rfloor\right) + 0.5 \right\rfloor$$

$$LHS = 2q + \left\lfloor \frac{r}{5} \right\rfloor$$

After simplifying RHS, we get,

$$RHS = 2q + \lfloor 0.1(r) + 0.5 \rfloor$$

We can now check each values of r from zero to nine, as it is the remainder of a number mod 10.

	LHS	RHS
r=0	2q	2q
r=1	2q	2q
r=2	2q	2q
r=3	2q	2q
r=4	2q	2q
r=5	2q+1	2q+1
r=6	2q+1	2q+1
r=7	2q+1	2q+1
r=8	2q+1	2q+1
r=9	2q+1	2q+1

With this we can say that for all branches of 10q, 10q+1, 10q+2, etc. the identity holds true.

Main takeaways

As the information was scattered all over the place, let us list them in order and also discover some new facts that were not in the above topics.

- Any number can be split into its constituent fractional and integer parts.
- The formula we made for extracting units digit renders useless for non integral values.

- Non integral denominators behave quite differently from integral denominators inside floor function.
- $\lfloor -x \rfloor = -\lceil x \rceil$
- $a \in \mathbb{Z} \Rightarrow \lfloor n + a \rfloor = \lfloor n \rfloor + a$ (this property led us to use that particular substitution in solution)
- While solving for infinite number of cases, we discovered some patterns(the groups of number having same remainder have same types of algebraic manipulations involved) which we solved individually.

Conclusion

This format of solving problems is elementary to number theory and discrete mathematics and it lays the ground work of how we understand the properties of numbers.