An Introduction to Chaos Through the Logistic Map

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1 Introduction

'Chaos' is used to describe disorder and confusion - a state in which we regularly find the universe. It suggests randomness. Certain natural phenomena, such as the famous 'three body problem', have tested the greatest mathematicians and physicists of our times. Randomness would offer a very easy answer - however, it remains deeply unsatisfactory. Chaos theory offers something more precise. Chaos theory studies systems that appear chaotic on the surface but are controlled by deterministic laws. Planned mayhem (in a way.) Whilst it has deep roots in philosophical ideas, chaos theory is predominantly a mathematical study - it is treated as a branch of mathematics involved in the study of non-linear dynamical systems which display high sensitivity to initial conditions.

2 Background: non-linear dynamical systems

Non-linear dynamical systems exhibit the properties of non-linearity and those of dynamical systems.

Non-linearity describes the property of systems where the change of the output is not proportional to the change of the input. Non-linear systems are defined mathematically by non-linear sets of equations e.g. $y = x^2$ and $y = x$

A dynamical system is a mathematical model that shows how a state changes over time according to a fixed rule such as a recurrence relationship or differential equations. Dynamical systems come in two flavours: continuous and discrete. This essay will focus on discrete dynamical systems.

2.1 Discrete Dynamical Systems

Discrete dynamical systems do not change over a continuous time frame, but the next state of the system is determined by the previous state. These systems are usually modelled by what are known as iterative maps.

An iterative map is a sequence such that $x_{n+1} = f(x_n)$ for some $f : \mathbf{R} \to \mathbf{R}$ Essentially, the output of one step is the input of the next step - a recurrence relationship. The logistic map is the most famous example of a discrete dynamical system and that is where our introduction to chaos truly begins.

3 The Logistic Map

$$x_{n+1} = rx_n(1 - x_n)$$

This model, popularised by the biologist Robert May in 1976, displays remarkable characteristics as the value of r is altered.
Despite its simple quadratic form, this equation is a deceptively complex example of chaos.
We take x_n to be a state of x and x_{n+1} to be the state of x after one step in time. Additionally, we take r to be a parameter symbolising the growth rate in the range $0 < r < 4$.
It is important to note that x is a normalised population in the interval [0,1] - a population as a fraction of the maximum e.g. a population at maximum capacity would have an x value of 1.
It would be correct to wonder why the growth rate parameter only goes up to 4. This restraint is mainly mathematical. Since we have defined x to be in the interval [0,1] to ensure the model makes biological sense the value of x cannot go beyond 1. Thus, there must be a maximum growth rate r in order for x to remain in the interval. This can be deciphered using differentiation.
The maximum output of the logistic map can be found by treating $x_n(1 - x_n)$ as function $f(x)$ which can be maximised.

$$f(x) = x(1 - x) \tag{1}$$

$$f(x) = x - x^2 \tag{2}$$

$$f'(x) = 1 - 2x \tag{3}$$

We can now maximise the function by finding a stationary point of the curve.

$$f'(x) = 0 \tag{4}$$

$$0 = 1 - 2x \to 2x = 1 \to x = \frac{1}{2} \tag{5}$$

plug $x = 0.5$ back into $f(x)$

$$f(\frac{1}{2}) = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4} \tag{6}$$

the maximum value of $\frac{1}{4}r$ should be less than 1

$$\frac{1}{4}r < 1 \to r < 4 \tag{7}$$

The way in which this sequence behaves highly depends on the growth rate parameter. At r values between $0 < r < 1$ the population dies out eventually. This case can be shown diagrammatically using cobweb diagrams - a visual way of displaying the behaviour of discrete dynamical systems. A cobweb diagram works by first drawing the curve $y = f(x)$ and the line $y = x$. After drawing a vertical line from x_0 to the curve, alternating horizontal and vertical movements between the curve and the diagonal line $y = x$ can be traced to show the behaviour of the recurrence relationship. For $0 < r < 1$, the following cobweb diagram can be simulated using Python to show how the population tends to 0 with more and more steps.

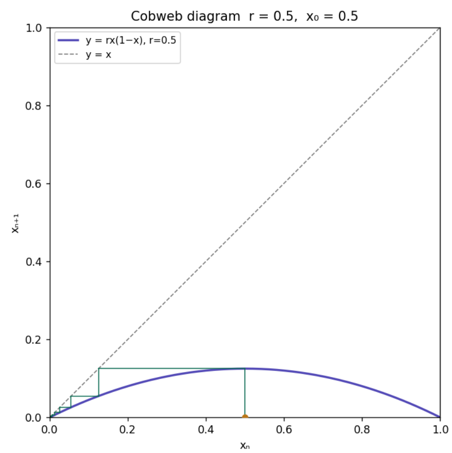


Figure 1: A cobweb diagram with $r = 0.5$ and $x_0 = 0.5$

The significance of this case is that it is related to fixed points and the stability of the logistic map.

3.1 Fixed points

Fixed points are values x^* such that for a map f, $f(x^*) = x^*$. In order to find the fixed points of the logistic map, we can solve the equation:

$$rx(1-x) = x \tag{8}$$

$$rx(1-x) - x = 0 \tag{9}$$

$$x(r(1-x) - 1) = 0 \tag{10}$$

if $x = 0$ then there will be a fixed point
additionally, if $r(1-x) - 1 = 0$, there will be another fixed point

$$r(1-x) = 1 \tag{11}$$

$$1 - x = \frac{1}{r} \tag{12}$$

$$x = 1 - \frac{1}{r} \tag{13}$$

Thus, the second fixed point of the logistic map can be given $x^* = 1 - \frac{1}{r}$

3.2 Stability and Equilibria

An equilibrium of a dynamical system is a value of state variables for which the state variables do not change - it can be considered as an interpretation of a fixed point. If we let the equilibrium be E then, $f(E) = E$. Whilst equilibrium is interesting, a more interesting concept, which is more tied into chaos, is stability - the behaviour of points around equilibria. Stability can be described based on whether a system converges upon an equilibria or whether it diverges: converging would suggest stability whilst diverging implies instability. In one dimension, the stability of the system at a specific state depends on the derivative.
Consider a fixed point x^*. Now, let ϵ_n be a small perturbation.
$x_{n+1} \approx f(x^* + \epsilon_n)$
By rearranging the differentiation from first principles formula:
$f'(x^*) = \lim_{\epsilon \to 0} \frac{f(x^*+\epsilon) - f(x^*)}{\epsilon}$
$f(x^*) + \epsilon f'(x^*) = f(x^* + \epsilon)$
Thus, $x_{n+1} \approx x^* + \epsilon_n f'(x^*)$
Additionally, we can say $\epsilon_n \approx \epsilon_0 [f'(x^*)]^n$ because we knew $x_{n+1} = x_n + \epsilon_{n+1}$
From this information, we can say that we can find out how ϵ_n changes over time and thus how x changes over time.
If $|f'(x^*)| < 1$ then over time $\epsilon \to 0$ because $\lim_{n \to \infty} [f'(x^*)]^n = 0$. This means that the orbit (an orbit is a sequence of states produced from iterating from a starting point) is stable. This sequence converges upon a fixed point.
On the other hand, if $|f'(x^*)| > 1$ then over time $\epsilon \to \infty$ because $\lim_{n \to \infty} [f'(x^*)]^n = \infty$. This sequence diverges and the orbit is unstable.
If $|f'(x^*)| = 1$, then we cannot conclude anything about the orbit. This suggests a bifurcation point on the logistic map, which we will talk about later.

Using the stability analysis we have just covered and the conditions $0 < r < 1$ we can now understand why all x values will move towards the fixed point $x^* = 0$. There is only one fixed point in the interval [0,1], 0, since $1 - \frac{1}{r}$ will be negative. $|f'(0)| = r < 1$ so the orbit is stable and for all values of x the population will converge upon extinction.

3.3 Stability for $1 < r < 3$?

Until now, we have been dealing with values for r between 0 and 1. For these values, there has only been one fixed point which is valid for the model used in the logistic map, 0. However, after $r = 1$, the other fixed point $x^* = 1 - \frac{1}{r}$ comes into play. We can address the stability of the system using the same method of

analysis we have just done.
$f'(x^*) = r(1-2x)$
$f'(1-\frac{1}{r}) = r(1-2+\frac{2}{r})$
$f'(1-\frac{1}{r}) = 2-r$
$|2-r| < 1$ is the stability interval
$-1 < 2-r < 1$
We can compute this to obtain the interval $1 < r < 3$. Therefore, we can tell that the map is stable for r values between 1 and 3.
Additionally, the other fixed point, $x^* = 0$ becomes unstable - we can call this a repelling fixed point.
This is because:

$$f'(x^*) = r(1-2x^*) \tag{14}$$

And since $x^* = 0$:

$$f'(x^*) = r < 1 \tag{15}$$

Therefore, since, $|f'(x^*)| = r > 1$, the fixed point $x^* = 0$ is unstable. On the contrary, we can call $x^* = 1-\frac{1}{r}$ an attracting fixed point (the system converges upon this fixed point.) This can be displayed using a cobweb diagram.

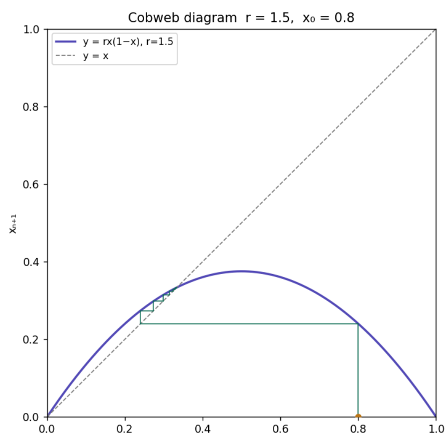


Figure 2: $r = 1.5$ and $x_0 = 0.8$. The system is stable when $1 < r < 3$

We can see that the system converges upon the stable fixed point $x^* = 1-\frac{1}{1.5} = \frac{1}{3}$ rather than the unstable, repelling fixed point $x^* = 0$.

3.4 The loss of stability; the real start of chaos

When $r = 3$,
$f'(0) = r > 1$ so the fixed point $x^* = 0$ is still unstable.
$f'(1-\frac{1}{r}) = r(1-2+\frac{2}{r}) = 2-r$
When $r = 3$, $2-3 = -1$, $|-1| = 1$ and thus:
$|f'(x^*)| = 1$

Neither of the fixed points are completely stable, and the logistic map starts to show off some distinct characteristics.

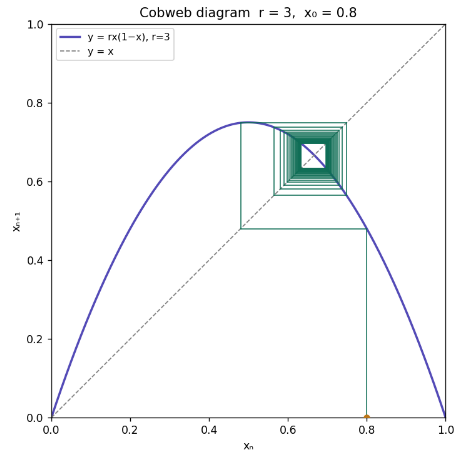


Figure 3: A cobweb diagram with $r = 3$

At $r = 3$, there is a bifurcation point. A bifurcation is defined as a qualitative change in a system's dynamics or long-term behaviour as a result of a variation of its parameters. At $r = 3$, a fixed point still exists, but the difference lies with the nearby points. If we return to the concept of perturbations with the formula $\epsilon_n \approx \epsilon_0[f'(x^*)]^n$ we can deduce what will happen to the system at $r = 3$ since we know that the value of $f'(x^*)$ is -1. Since this is negative, after being raised to increasing powers the direction of the perturbation will switch every increment in n whilst the magnitude of the perturbation will remain the same. This verges on the much greater loss of stability as r increases beyond 3. For $r > 3$, $f'(x^*) < -1$. This means that small perturbations from x^* are amplified and thus, the fixed point become unstable. After each iteration, the perturbation flips sign and grows. This gives way to a period-2 orbit (this is where the first period-doubling bifurcation occurs.) Instead of settling to one state, the system settles to 2 states. A period-2 orbit is one which takes 2 steps to return to its initial position (before the 2 steps) - after a few iterations, the period-2 orbit should form. If we give the two points the labels p and q, we can say $f(p) = q$ and conversely, $f(q) = p$.
It can be said that $f^2(p) = p$
For $3 < r < 1+\sqrt{6}$, the period-2 cycle is stable. By the same mechanism as before, at $r = 1 + \sqrt{6}$ there is another period-doubling bifurcation. Subsequently, the period doubles. A period-4 orbit is born in which the system returns to the same state every 4 steps - the system settles to 4 distinct values of x.
Throughout the range $3 < r < 4$, more period doubling bifurcations occur. We can observe the behaviour of the values of r at which these bifurcations occur.

Let r_n be the value of r at which the n^{th} bifurcation happens.

Bifurcation	Period of orbit	r_n	Gap between consecutive r_n values
1	2	3.0000	-
2	4	3.4495	0.4495
3	8	3.5441	0.0946
4	16	3.5644	0.0203
5	32	3.5688	0.0044

Table 1: Bifurcations

Computer experiments can reveal that $r_\infty \approx 3.569946$: infinitely many bifurcations occur at this value of r. Beyond this value of r, there is an onset of chaos. In this scenario, chaos mainly means two things: aperiodicity (not repeating the same value of x) and sensitivity to initial conditions.

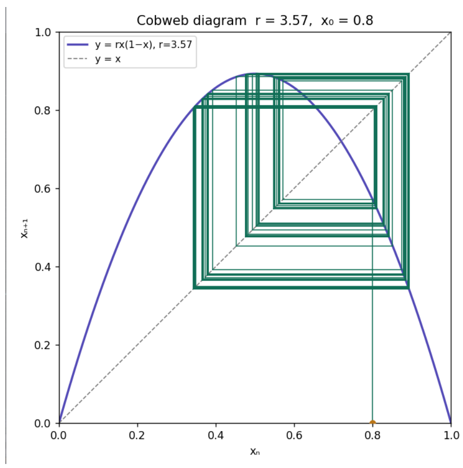


Figure 4: 50 steps of a cobweb diagram of the logistic map

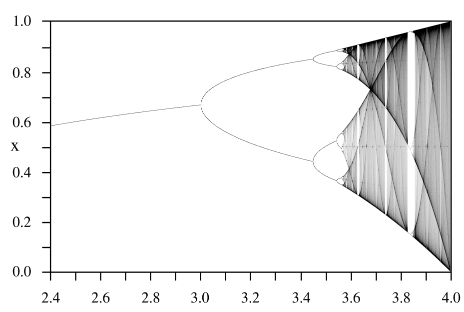
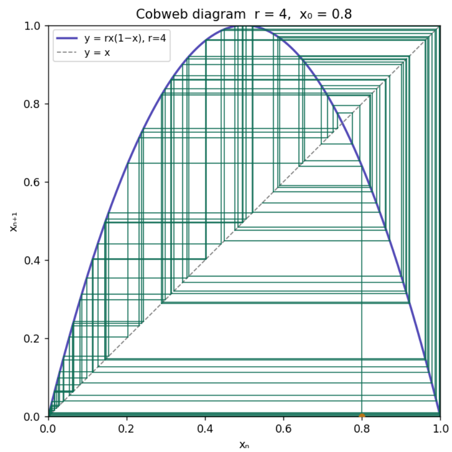


Figure 5: A visualisation of bifurcations

At $r = 4$, the map can be judged to be 'full' - chaos is most prevalent and the full interval is covered.



3.5 Lyapunov exponent

I mentioned that at $r = 4$, chaos is most prevalent. This is a qualitative remark - the Lyapunov exponent provides a quantitative approach to measuring chaos. The Lyapunov exponent measures the rate at which nearby trajectories diverge or converge, characterising the sensitivity of the system to initial conditions. At $r = 4$, the Lyapunov exponent is at its maximum value: $ln(2)$. The Lyapunov exponent is given by a difficult equation of the form:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \Sigma_{i=0}^{n-1} ln(r(1 - 2x_i)) \tag{16}$$

3.6 The Universality of Feigenbaum

It is fascinating to take note of a constant which appears throughout the logistic map and all other non-linear dynamical systems which approach chaos through period-doubling bifurcations.
The Feigenbaum constant, δ, is defined as the limiting ratio between successive bifurcation intervals.

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669201609 \tag{17}$$

From the table of the values for r_n above, by dividing the previous gap by the next gap, the way in which the ratio converges to δ can be observed.

	Gap $r_n - r_{n-1}$	Gap $r_{n+1} - r_n$	Ratio
$r_n = 2$	0.4495	0.0946	4.75...
$r_n = 3$	0.0946	0.0203	4.66...
$r_n = 4$	0.0203	0.0044	4.61...
$r_n = 5$	0.0044	0.00093	4.73...
$r_n = 6$	0.00093	0.000199	4.67..

Table 2: The ratios converge upon δ

This is converging on the Feigenbaum constant.

This ratio can be found by observing the period-doubling bifurcations in other maps as well, such as the sine map, $rsin(\pi x)$. This is shocking because the algebraic and geometric nature of the sine map contrasts with the logistic map (one is a polynomial whilst the other is trigonometric) but the Feigenbaum constant still arises, δ, because any map with a single hump will produce this. Universality, therefore, is true for chaos - for any system with period-doubling bifurcations chaos arises in the same route, with the same ratio.

4 Conclusion

Chaos theory has been something which has been scrutinised and developed over the past century by our greatest minds including the likes of Edward Lorenz and Henri Poincaré. Lorenz famously delivered a lecture in 1972 titled 'Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?' This gave way to what is known as the butterfly effect and reminds us of the main concept behind chaos theory. It is not randomness, but sensitivity to initial conditions, that is at the forefront of chaos theory. This has been shown through our study of the logistic map: after the system turned chaotic, small perturbations from equilibrium amplify and result in a system which appears to be out of control. The beauty of it all is that there is math behind it all - the universal constant, δ, consolidates this. For any smooth map with one turning point, there is one constant behind the chaos - a universal connector between biological population modelling, electric circuits and fluid dynamics.

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