Analysing Analysis

introduction

In order to establish a sense of ease among non-specialists i shall start this essay with a few familiar names within the history of mathematics: **Issac Newton, Leonhard Euler, Bernhard Riemann, Henri Poincaré**. But **what** do these names have in common (other than being mathematicians held in an almost legendary regard)? If it hasn't been made obvious as of yet by the title of this essay, these figures have all been attributed to the development and influence of both real and complex analysis. But what exactly is mathematical analysis? Where is it used? How was it developed? These are all questions i plan to answer while conducting this analysis on analysis (**pun intended**).

Naturally if you are already familiar with this area of mathematics, as most math graduates are, this essay could be of less use to you, however even so i intend to delve into the origins and intricacies of analysis such that (**or as some would notate ∍**) even those who have studied analysis can learn at least a few new things.

The first topic i aim to tackle within this essay is the origins of analysis itself, including it's history, progenitors and uses early in its lifetime; and so i shall end my introduction here and begin my analysis in analysis.

origins

Now although the formal developments of analysis can be seen to date back to the 17thC's scientific revolution (**which funnily enough is also credited to the development of calculus, which is largely considered a result of analysis**) , many of its ideas can be traced back to early mathematicians from eras such as : ancient Greek, ancient Indian and ancient Egyptian. But how was analysis used in these early times? Surely it wasn't as advanced as modern analysis?

To provide an adequate example, the topic of infinite geometric sums were touched on by Zeno of Elea (for those unaware of this figure, Zeno was a pre-Socratic Greek philosopher and student of **Parmenides**, who held that reality is singular, unchanging, and that motion and plurality are illusions. But as to not excessively analyse the history of ancient Greece, i will briefly end this description) in a well-known example of the famous story of Achillies and the tortoise.

I'm sure **99.9...% (or well..100%)** of you are already familiar with this ancient tale, however today we look at it through a mathematical lens.

Achillies, the tortoise and the paradox of Zeno

As the story goes, Achilles; the greatest warrior and fastest runner in Greek mythology challenges a tortoise to a race. Feeling generous, he gives it a head start of around 100 metres.

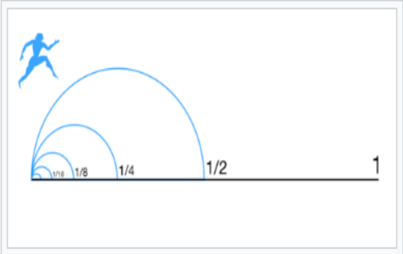
The race erupts. Achilles sprints toward where the tortoise started. But by the time he reaches that point, *the tortoise has shuffled forward a miniscule 10 metres*. So Achilles runs to ***that*** point. But again, by the time he arrives, the tortoise has moved on an even smaller 1 metre. Achilles reaches that point. But to his dismay, the tortoise has moved again a tiny bit further still.

This continues **forever**. Every time Achilles reaches where the tortoise *was*, it has crept fractionally ahead. There are always infinitely many such intervals to close, with the intervals becoming so small, that it eventually appears as if neither Achillies nor the tortoise are moving at all!

Now while this may seem like an old children's tale, we can now view it as Zeno's paradox truly being the concepts of limits waiting to be discovered; **a question existing over 2000 years before its answer**. We can establish the paradox's relation to analysis through the concept of 'limits' and modelling this paradox mathematically which i will do my best to explain.

If we let the distances Achilles must cover form the sequence:

100, 10, 1, 0.1, 0.01... (following a pattern of becoming decreasingly small)



the total distance becomes the infinite sum:

100, 10, 1, 0.1, 0.01...= $\sum_{n=0}^{\infty} 100 \cdot \left(\frac{1}{10}\right)^{n} = \frac{100}{1-\frac{1}{10}} = \frac{1000}{9} \approx 111.1$

(The sum on the left hand side is known as a convergent sum, as it's partial sums get **closer and closer and closer...** to a fixed finite value as more sums are added, never actually reaching this value. This process of getting 'infinitely close' is exactly what a limit within analysis describes!)

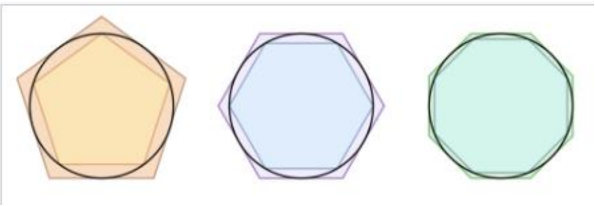
-I have attempted a simple visualisation of this specific paradox which can be found via the link https://www.desmos.com/calculator/qyxxe3rznc , adjust the k slider to watch as the graph reaches but does not touch the established limit.-

And so our **analysis** on the tale of Achillies and the tortoise has come to an end **(although their race won't seem to end any time soon)**, now we will continue our look into analysis' Greek past.

Eudoxus' Exhaustion

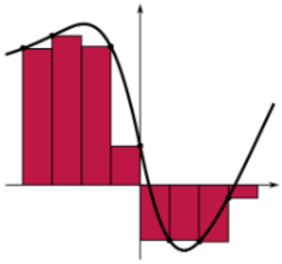
Following the theme of ancient Greek mathematics and the origins of ideas contributing to analysis, we arrive to Eudoxus of Cnidus. A polymath working in fields such as astronomy, mathematics, law and medicine who is also credited for the development of Antiphon's 'Method Of Exhaustion' which acts as both a precursor to integral calculus and the concept of limits, the latter acting as the foundation of analysis, solidifying this method of exhaustion as massively important when considering contributions to modern day analysis. **But what exactly is the method of exhaustion?**

Put simply, the method of exhaustion is used to find the area, volume or some other quantity of a shape by inscribing it in a sequence of polygons whose areas converge to the area of the containing shape. Now less on Eudoxus and more on exhaustion, which was also used by fellow Greek polymath Archimedes of Syracuse to approximate the area inside a circle, which can be visualised as such.



Here we can clearly see the method of exhaustion being used to compute the area of a circle. Furthermore, this can also prove the area of a circle to be πr^2 through noting that as the number of sides of the polygon increases, the area of said polygon creeps closer and closer to that of the circle although never reaching the value exactly. Through comparing this area to the square of the radius of the circle, it can be shown that the area of the circle is πr^2 where pi is the ratio of the circumference to the diameter, providing a slight hint into the more rigorous, proof-based side of analysis that many are familiar with. A more modern take on Eudoxus' exhaustion within the context of analysis appears when approximating an integral with a Riemann sum; with a Riemann sum being simply put, a way of approximating the area under a curve by dividing it into thin rectangles, adding up their areas, and taking the limit as the rectangles get infinitely thin which gives the exact integral.

This can easily be visualised as such.



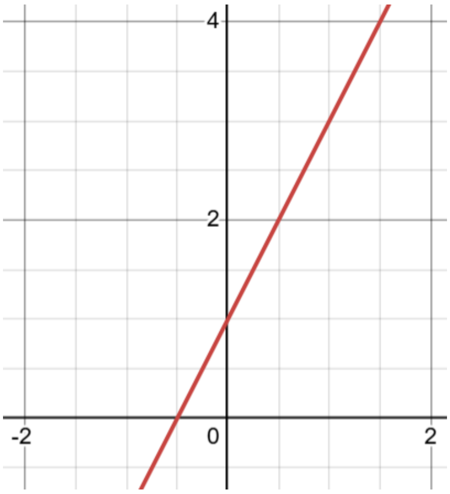
Although this idea may seem relatively simple at first, it acts as a foundation for all the theory of integral calculus, seeing practical applications from physics and economics to medical imaging and archaeology; showing the almost **limitless** applications of analysis and calculus.

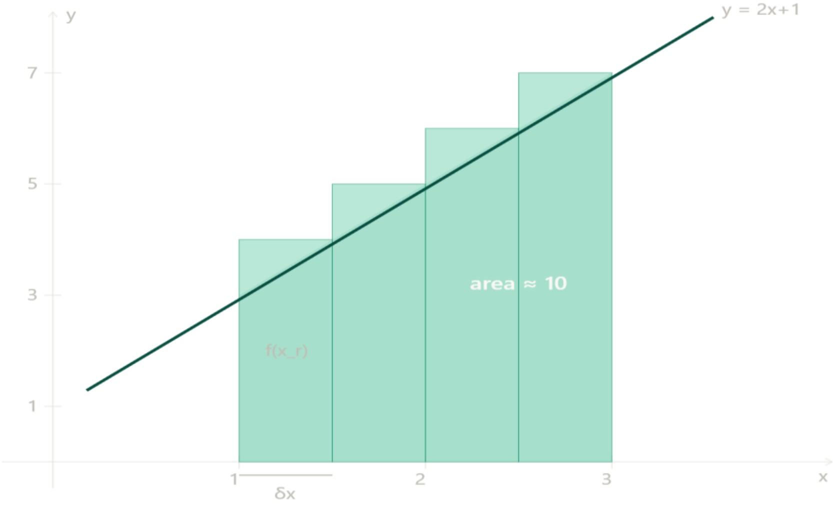
Example of a Riemann sum question

As we creep closer towards the end of this essay, i would like to include an example of an analysis-based question, which will be to…

evaluate the simple integral $\int_1^3 (2x + 1)\,dx$, which is graphed is such

Within this specific example, we are simple being asked to find the area under the visible graph between x=1 and x=3, however instead of integrating directly we find it by summing rectangles.





Here we can visualise the individual rectangles making up our approximation for the area of the graph when n=4, established between the limits of x=1 and 3 as aforementioned. However, each strip slightly overshoots the true area, but as n increases and tends to infinity this overshoot also decreases, and the strips perfectly summate to the area of the lines given the respective limits of x; With each strip having a width of $\delta x = \frac{2}{n}$.

Moreover, let each strip have a height equal to the function value at its right edge, being$f\left(1 + \frac{2r}{n}\right) = 3 + \frac{4r}{n}$

Then we simply sum all the rectangles we have, leading to the equation$Sn = \frac{2}{n} = \sum_{r=1}^{n}\left(3 + \frac{4r}{n}\right) = 6 + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} = 6 + \frac{4(n+1)}{n}$

Finally, we take the limit as n tending towards infinity, as we have learned the rectangles will have to become infinitely thin to fill the area to an exact value, leaving us with $n \to \infty lim\, 6 + \frac{4(n+1)}{n} = 6 + 4 = 10$

I hope this example can suffice as a simple introduction to integrals with Riemann sums, however, don't be misled; as a non-specialist aimed example such as this is hardly the tip of the iceberg of the beautiful topic **analysis**.

Example question 2

With this example i aim to target the more proof heavy side of analysis that many are acquainted with and is also my personal favourite part of the topic that i have studied so far. The question(s) i have decided on are a couple of questions from chapter one of

Walter Rudin's "Principles of Mathematical Analysis", requiring next to no prerequisite knowledge on the subject and are as follows...

1.If r is rational (and non-zero) and x is irrational, prove that r+x and rx are irrational.

2.prove there is no rational number whose square is 12

Q1

For this question, i will prove by contradiction; meaning assuming the opposite of what i would like to prove and then showing the consequences of this to be impossible.

-let r+x and rx be = to some rational number y which will be denoted $\frac{k}{z}$

- Since r is rational, denote r as $\frac{p}{q}$ such that (or ∋) $\frac{p}{q}$+x and ($\frac{p}{q}$)(x)=y

-Tackling $\frac{p}{q}$+x=$\frac{k}{z}$ first, we can manipulate this equation into the form x=$\frac{k}{z}$ - $\frac{p}{q}$

-This is a contradiction as an irrational number cannot be equal to the difference between two rational numbers, thus r+x is irrational

Q2

Similarly, here i will prove via contradiction, letting rx=y

-First set ($\frac{p}{q}$)x=$\frac{k}{z}$

-solve for x, leaving us with x=$\frac{pz}{qk}$

-Since the division of two rational numbers leads to a rational result, we know that $\frac{pz}{qk}$ is rational.

-Thus x=$\frac{pz}{qk}$ is contradictory and so rx cannot be rational

Similar to Riemann example, these serve only as introductory level questions to proofs within analysis; excluding any tricky theorems or pre-requisites to accommodate non-specialists. However, i do greatly recommend **Walter Rudin's "Principles of Mathematical Analysis"** to anyone wanting to gain even an introductory level understanding of the subject, as i personally believe it guides readers that may be new to analysis quite well, establishing ground rules to the topic which are then built up in an extraordinary fashion starting from as soon as the first chapter! However sadly i cannot delve into the more rigorous proofs and questions, as by now I'm at a loss for words.

Conclusion

Sadly, as this essay reaches it's **limit**; it must come to a close. However i have great hopes that all readers of this essay will learn at least one or two new things, whether it be about the 2000-year-old race, or great polymaths of the past.
Since my motivation for the topic of this essay came from my personal studies on professor Rudin's textbook, i felt it completely necessary to reference it at least once within this essay, not only to introduce people to his work, but to analysis as a whole, which I initially decided to look at as it is a foundational prerequisite to a much more advanced topic i wish to study much later down the line; that being **modular forms**.
As to not drag this essay towards it's **limit** (this joke must be getting old) i must conclude by thanking not only the reader, but also Professor Tom Crawford allowing students and adults alike to express and explore their favourite topics! I can only guarantee the next time i write a math based essay, it must be on a topic related to **analysis**.

References

- https://en.wikipedia.org/wiki/Riemann_sum
- https://en.wikipedia.org/wiki/Eudoxus_of_Cnidus#Mathematics
- https://en.wikipedia.org/wiki/Method_of_exhaustion
- https://en.wikipedia.org/wiki/Mathematical_analysis