The Game of Nim

1.Introduction

Nim is one of the central examples studied in Combinatorial Game Theory, the branch of mathematics that analyses strategy games using mathematical reasoning. Although the rules are extremely simple, the strategy behind Nim involves ideas such as binary numbers, invariants, and logical reasoning. In fact, Nim was one of the first games ever “solved" completely by mathematicians. It is said to be one of most ancient games in the world, originating from China, with the current name ‘Nim’ coming from the German word “nimm”, which means “take”.

The Rules of Nim

The Game of Nim is played with several piles of stones, for example. The rules are very simple:

1. The game begins with several piles containing some number of stones.
2. Two players take turns.
3. On each turn, a player chooses **one pile** and removes **any number of stones** from that pile.
4. The player who removes the **last stone** wins.

A typical starting position might look like: (3,4,5), meaning there are three piles containing 3, 4, and 5 stones respectively.

Possible moves include:
(3,4,5) to (3,4,2) (removing three stones from the third pile)

(3,4,5) to (3,1,5) (removing three stones from the second pile)

The only restriction is that a player must change exactly one pile each turn.

Therefore, the question that arouses is:

Can we determine who will win from a given starting position if both players play optimally?

In 1901-1902, Harvard professor Charles Leonard Bouton proved that every possible position in the game can be classified as either winning or losing if both players play optimally.

Winning and Losing Positions

To analyse Nim mathematically, it is useful to classify positions into two categories:

- **Winning position**: the player whose turn it can force a win.
- **Losing position**: the opponent can force a win.

The goal is to identify which positions are winning and which are losing

2.Patterns

Case one

If there is only one pile that we note (n), the situation is obvious. The player whose starts can remove all the objects and win immediately.

So, every position with a single non-empty pile is a **winning position**.

Case two

Now consider two piles (a,b)

We first notice that positions where the two piles are equal,
(1,1),(2,2),(3,3)
are losing positions.

The reason is simple: the second player can copy the first player’s move in the other pile. If the first player removes (k) objects from one pile, the second player removes (k) from the other. Eventually the first player is forced to leave one object for the opponent.

So, with two piles, the strategy is clear: <u>leave equal piles for your opponent whenever possible.</u>

But once we introduce a third pile, things become much less obvious.

More than Two Piles

When extending Nim beyond one or two piles, the structure of the game becomes seemingly much more complex. The pattern recognition we just did for one and two piles is no longer possible for analysing the winning and losing positions. Instead, we treat piles with N≥3 as a general framework. Suppose we have three or more piles with sizes $a_1, a_2, \ldots, a_n$. The challenge is to determine whether a given configuration is winning or losing and, if winning, how to move to a losing configuration for the opponent.

An important observation simplifies the analysis: each pile behaves independently in terms of allowed moves (because a move only affects the number of stones in one pile only), yet the underlying link between the multiple piles determines the outcome.

Nim-Sum

Let's start by taking the example of the configuration (1,2,3). We notice that the "copying method" that worked earlier is ineffective here. So, to determine whether this is a winning position or not, we are looking for a link between general winning positions and the position itself. And this link lies in the binary representation of these asymmetric configurations:

So, we try a different approach: represent the numbers in binary.

$$1 = 001 \qquad 2 = 010 \qquad 3 = 011$$

But now our perspective changes, we write the original position in binary forms in columns:

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$$

Now we look by column by column, so for each power of 2 in the binary forms obtained.

We then count how many 1s appear in each column:

- Rightmost column: $1 + 0 + 1 = 2$ (even)
- Middle column: $0 + 1 + 1 = 2$ (even)
- Leftmost column: $0 + 0 + 0 = 0$ (even)

So, every column has an even number of 1s. For now, let us express this situation as **balanced.**

Now take one possible move and see what changes.

$$\mathbf{(1, 2, 3) \to (0, 2, 3)}$$

Binary:

$$0 = 000, 2 = 010, 3 = 011$$

Columns:

- Rightmost: $0 + 0 + 1 = 1$ (odd)
- Middle: $0 + 1 + 1 = 2$ (even)
- Leftmost: $0 + 0 + 0 = 0$ (even)

Now there is an odd column, an **imbalance** appears.

By trying other different moves from our “balanced” positions we notice that the next position is always “imbalanced”. If all columns are even, at least one is odd after a move.

We can also check if the opposite direction is possible:

We take the “imbalanced” position (0,2,3) and we make a move by removing a stone from the last pile, which gives us (0,2,2).

In binary numbers this is:

000 010 010

Columns:

- Rightmost: $0 + 0 + 0 = 0$ (even)
- Middle: $0 + 1 + 1 = 2$ (even)
- Leftmost: $0 + 0 + 0 = 0$ (even)

Thus, the balance is restored, and it could be possible to go from an “imbalanced” position to a “balanced” one.

But how can we link the parity of each column independently a winning/losing position?

This is the point of the Nim-sum or XOR sum: we add the binary form of the pile sizes column by column.

Formally, let the pile sizes be $a_1, a_2, \dots, a_n \in \mathbb{N}$. The Nim-sum is defined as the bitwise XOR of pile sizes:

$$S = a_1 \oplus a_2 \oplus \cdots \oplus a_n.$$

Lemma 1

Part 1 : A position is a winning position (a W-position after the move) if $S = 0$

Part 2: A position is a losing position (L-position after the move) if $S \neq 0$.

Proof

Part 1:

Let us consider a game state that we denote as $(a_1, \dots, a_k, \dots, a_n)$

Suppose $S = 0$. Consider any move:

$$(a_1, \dots, a_k, \dots, a_n) \ \rightarrow \ (a_1, \dots, a'_k, \dots, a_n), \text{with } 0 \leq a'_k < a_k \text{ and } 1 \leq \text{k} \leq \text{n}$$

We first notice that $\mathrm{b} \oplus \mathrm{b} = 0$

Let

$$S' = a_1 \oplus \cdots \oplus a'_k \oplus \cdots \oplus a_n$$

Then

$$S' = S \oplus a_k \oplus a'_k = 0 \oplus a_k \oplus a'_k = a_k \oplus a'_k.$$

Since $a'_k \neq a_k$, we have $a_k \oplus a'_k \neq 0$, hence $S' \neq 0$. Therefore,

$$S = 0 \implies \forall \text{ moves, } S' \neq 0,$$

so every move leads to an L-position.

Part 2:

Suppose $S \neq 0$. Let 2^m be the highest power of 2 appearing in the binary expansion of S (the most significant bit where S has a 1. Then there exists some a_k whose binary expansion also has a 1 in position m. Define

$$a'_k = a_k \oplus S.$$

Then $a'_k < a_k$ (since the leading 1 at position m is removed), so this is a legal move. Let

$$S' = a_1 \oplus \cdots \oplus a'_k \oplus \cdots \oplus a_n.$$

Then

$$S' = S \oplus a_k \oplus a'_k = S \oplus a_k \oplus (a_k \oplus S) = 0.$$

Thus,

$$S \neq 0 \implies S' = 0.$$

Hence,

$$S = 0 \iff W\text{-position} \qquad S \neq 0 \iff \text{L-position}. \qquad \blacksquare$$

3.Sprague-Grundy theorem

The result of Nim led to a generalisation for any impartial game (game where both players have the same allowed moves, with the only constraint being the position in the game). Essentially, the Sprague-Grundy theorem states that any such game is equivalent to a game of Nim with one pile, such as the one we saw at the start. This theorem is the result of separate discoveries by Roland Sprague in 1935 and Patrick Michael Grundy in 1939.

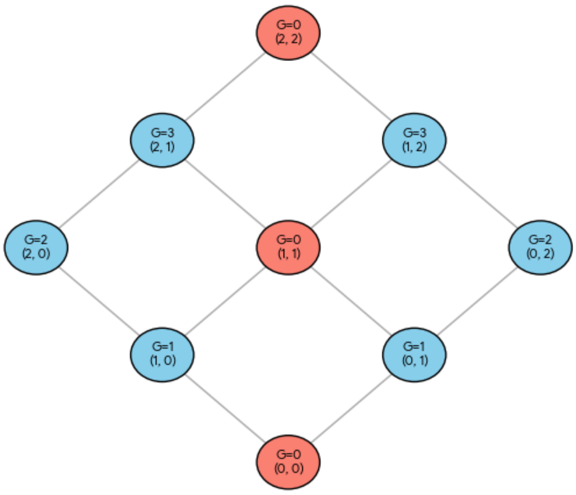
Nimber

Let us consider an impartial game A, where from any position there is a set of positions reachable in one move. The key idea is to assign to each position a number $g(A) \in \mathbb{N}$, called the Grundy number, or "Nimber", which measures the equivalent size of a Nim pile that behaves identically to our game A.

This value is defined recursively using the rule

$$g(A) = \text{mex}\{\ g(A') \mid A' \text{ is reachable from } A\ \},$$

Grundy Values in State Graph (Nim 2,2)



where mex (minimum excluded value) is the smallest non-negative integer not contained in the given set.

For example:

- If A has no moves, then the set is empty, so

$$g(A) = \text{mex}\ (\emptyset) = 0.$$

- If the options of G have Grundy values $\{0,1,2\}$, then

$$g(A) = 3.$$

- If the options have values $\{1,2\}$, then

$$g(A) = 0.$$

This definition ensures a crucial property:

$$g(A) = 0 \iff A \text{ is a } W\text{-position.}$$

The result is that impartial game behaves like a Nim pile of size $g(A)$. In other words,

$$A \rightarrow \text{Nim heap of size } g(A)$$

meaning they are strategically equivalent under optimal play.

Combined games

Now consider a sum of independent subtraction games (another type of impartial game but different to Nim).

Ex: each player can remove 1,2, or 3 stones from a total of ten:

$$G = G_1 + G_2 + \cdots + G_n,$$

Where G_1, G_2 ... are entirely separate subtraction games.

A move consists of choosing one component G_i and making a move only in that component (likewise to picking one of many piles for Nim). The Sprague–Grundy theorem states that:

$$g(G) = g(G_1) \oplus g(G_2) \oplus \cdots \oplus g(G_n).$$

This is the same operation as the Nim-sum. Therefore, even though each G_i may be a completely different game, their combination reduces to a single Nim heap whose size is the XOR of their Grundy numbers/Nimbers.

Hence, the winning condition becomes:

$$g(G_1) \oplus g(G_2) \oplus \cdots \oplus g(G_n) = 0 \iff \mathcal{P}\text{-position.}$$

To see why this works, note that any move changes exactly one component:

$$G_i \to G_i',$$

so the total Grundy value changes as

$$g(G) \to g(G) \oplus g(G_i) \oplus g(G_i').$$

This XOR sum of Grundy values brings us back to the Nim proof:

By Lemma 1, without loss of generality, one can always choose a move that makes the total XOR equal to zero when it is initially non-zero.

If we consider Nim, we have: a pile a of size k has options of sizes $0,1,\dots,k-1$, so

$$g(a) = \text{mex}\,\{0,1,\dots,k-1\} = k.$$

Thus, recovering the original Nim-sum of our Nim game denoted A:

$$g(A) = a_1 \oplus a_2 \oplus \cdots \oplus a_n,$$

4.Other connections of Nim

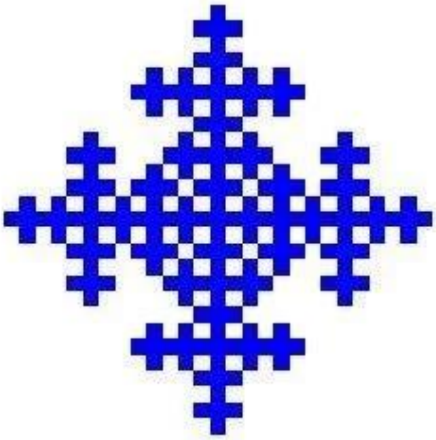
An unexpected connection arises with the Ulam–Warburton automaton, a cellular automaton defined on a square grid. Starting from a single live cell, new cells are added iteratively: at each step, a cell is added if it has exactly one live neighbour. This generates a fractal pattern.

The link to Nim and Grundy theory emerges through the shared role of binary form and parity. In Nim, the key operation is the XOR sum:

$$a_1 \oplus a_2 \oplus \cdots \oplus a_n$$

which is equivalent to addition in $(\mathbb{Z}_2)^k$.

Each bit evolves independently, and the overall behaviour is determined by parity. The growth of the Ulam–Warburton automaton can be analysed in terms of binary representations of coordinates. Whether a cell appears at a given stage depends on parity constraints: the binary parity of each generation in the fractal pattern connects to XOR sum.



Ulam-Warburton cellular automaton

Additionally, the recursive definition

$$g(G) = \text{mex}\,\{g(G')\}$$

and the automaton rule ("a cell is born if it has exactly one neighbour") are both local rules that generate global complexity. In Nim, the recursion builds a complete classification of game states; in the automaton, it generates an infinite geometric object.

These two match to each other, as cell births in the automaton directly correspond to positions of Grundy values $\neq 0$. Therefore the game of Nim can be viewed as a cellular automaton.

Thus, both systems can be viewed as processes over binary states:

Nim: $\oplus\, g(G_i)$ Automaton: growth governed by parity of neighbours.

5.Conclusion

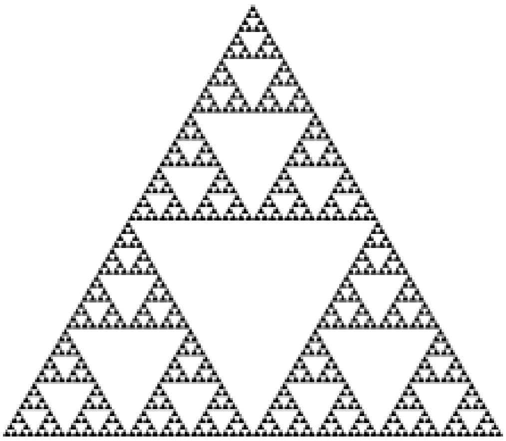
In conclusion, the simple scenario with stones and piles led us to discover much more complex ideas such as binary forms, XOR sum and especially the Sprague-Grundy theorem, which applies to all impartial games

There are many other impartial games which can be analysed thanks to the Sprague-Grundy theorem.

Some of these include:

- **21 Game**: Players start with 21 items and can take items (usually 1-3). The person who takes the last item loses.
- **Green Hackenbush**: Players cut edges of connected segments and last move wins.
- **Kayles**: players knock down adjacent pins (1-2) and last move wins.

Another possible illustration like the automaton is the Sierpinski triangle, whose structure aligns with the positions in an impartial game with Grundy value 0.



Sierpinski Triangle

What I find so interesting about the Nim Game is the idea that a determined outcome is possible if the player understands the more complex, underlying factors that we have covered on this essay. Unlike other common two player games such as chess or checkers, Nim is impartial, and therefore can be solved, which leads to a whole field of mathematics and computer science: combinatorial game theory.