

# Andrew Cheng - 2026 math essay competition

My favorite thing about math is how you can just screw around and from that find out new things. Playing around with numbers grants you amazing satisfying properties in every field. When I was just in 6th grade, for example, I discovered an intriguing property that, although I know is nowhere innovative, was discovered independently by me, and thus is something I adore.

I had some free line paper, and so, bored by the class, I began writing the powers of 2. Every time I wrote a new one, I made sure to move the ones place one slot over. So, the first few terms of this was the following:

2  
04  
008  
0016  
00032  
000064  
0000128  
00000256  
000000512  
0000001024  
...And so on

After I did this for a while, I added up each of the terms, ending with 24999999.... I noticed that, if I put a decimal before the start of each term, I'd get  $.2 + .04 + .008... = .24999...$ , which is  $\frac{1}{4}$ . I also noticed that the way of making each new term was multiplying by 2, and then dividing by 10, which ends as a division by five. So, that implied to me that adding powers of one fifth made it go to one fourth.

However, that was not at all rigorous. It's a good estimate, but not at all precise. So, when this contest was announced, I revisited my discovery, and decided not to only prove this true for  $\frac{1}{5}$ , but for all fractions  $\frac{1}{x}$ .

I started with a construction, some shape with area 1. After that, I colored in  $\frac{x-1}{x}$  of the shape, leaving  $\frac{1}{x}$  left. I repeated this process on the smaller figure, coloring in a total of  $\frac{1}{x} \cdot \frac{x-1}{x}$  and leaving  $\frac{1}{x} \cdot \frac{1}{x}$ . Repeating this again gets  $\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{x-1}{x}$  and  $\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x}$  respectively, and this can be extended forever.

An example using  $x=3$



So, at infinity it'd be

$$1 = \frac{x-1}{x} + \frac{1}{x} \cdot \frac{x-1}{x} + \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{x-1}{x} \dots$$

Simplifying the terms, it becomes

$$1 = \frac{x-1}{x} + \frac{x-1}{x^2} + \frac{x-1}{x^3} \dots$$

Diving each side by  $x-1$ , we get

$$\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} \dots$$

Which can be expressed as

$$\sum_{i=1}^{\infty} \frac{1}{x^i} = \frac{1}{x-1} \quad \text{given } x > 1$$

But of course, we can't stop here. Sure, we know it for the numerator being 1, but what about any arbitrary numerator? This can be expressed as

$$\sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i$$

So, how do we solve this? We can start by multiplying by  $1^i$ , that will always equal 1 and won't change the answer. So:

$$\sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i = \sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i 1^i$$

After this, we can replace 1 with something equaling 1, like  $(1/m)/(1/m)$ . This gives the equation

$$\sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i 1^i = \sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i \left(\frac{1/m}{1/m}\right)^i$$

Now, because of the way these fractions work out, that must mean that

$$\sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i \left(\frac{1}{\frac{m}{n}}\right)^i = \sum_{i=1}^{\infty} \left(\frac{m \cdot \frac{1}{m}}{n \cdot \frac{1}{m}}\right)^i$$

*Simplifying each side of the fraction, we get that*

$$\sum_{i=1}^{\infty} \left(\frac{m \cdot \frac{1}{m}}{n \cdot \frac{1}{m}}\right)^i = \sum_{i=1}^{\infty} \left(\frac{1}{\frac{n}{m}}\right)^i$$

*And since 1 to the power of anything is 1, this means that*

$$\sum_{i=1}^{\infty} \left(\frac{1}{\frac{n}{m}}\right)^i = \sum_{i=1}^{\infty} \frac{1}{\left(\frac{n}{m}\right)^i}$$

*But wait! This equation is the same as the previous one we've proved, except with  $x$  being replaced with  $\frac{n}{m}$ . Since we know how to solve that equation, then that must mean that:*

$$\sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i = \frac{1}{\frac{n}{m} - 1} \quad \text{given } n > m$$

*And still, we shan't stop here. We've solved the case if it starts at 1, but what about any arbitrary number? So, can we solve*

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i$$

*Now, take any  $j$*

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i = \frac{m^j}{n^j} + \frac{m^{j+1}}{n^{j+1}} + \frac{m^{j+2}}{n^{j+2}} \dots$$

*Now, each term  $j$  can be replaced by  $j-1$  and then add one more than what was there normally. So, it would look like*

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i = \frac{m^{1+j-1}}{n^{1+j-1}} + \frac{m^{2+j-1}}{n^{2+j-1}} + \frac{m^{3+j-1}}{n^{3+j-1}} \dots$$

Separating each exponent into its own fraction, we have

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i = \left(\frac{m^{j-1}}{n^{j-1}}\right) \frac{m^1}{n^1} + \left(\frac{m^{j-1}}{n^{j-1}}\right) \frac{m^2}{n^2} + \left(\frac{m^{j-1}}{n^{j-1}}\right) \frac{m^3}{n^3} \dots$$

Each of these terms has a factor of  $\frac{m^{j-1}}{n^{j-1}}$ , so separating that gives us

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i = \left(\frac{m^{j-1}}{n^{j-1}}\right) \left(\frac{m^1}{n^1} + \frac{m^2}{n^2} + \frac{m^3}{n^3} \dots\right)$$

Hey, doesn't the second part of that term look familiar? Yes it does! It's the expanded form of our earlier equation! Replacing it with that, we have

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i = \left(\frac{m^{j-1}}{n^{j-1}}\right) \left(\sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i\right)$$

Now, we know what  $\sum_{i=1}^{\infty} \left(\frac{m}{n}\right)^i$  is, remember? Replugging that, we get

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i = \left(\frac{m^{j-1}}{n^{j-1}}\right) \left(\frac{1}{\frac{n}{m}-1}\right) \quad \text{given } n > m$$

Because of the way fractions work, it's to be said that therefore, in general,

$$\sum_{i=j}^{\infty} \left(\frac{m}{n}\right)^i = \left(\frac{m^{j-1}}{\frac{n^j}{m} - n^{j-1}}\right) \quad \text{given } n > m$$