THE IMPOSSIBLE SPIRAL: How a differential equation scored for Brazil

1. The Pearl of Wisdom:

I was in the 10th grade, and I remember my extremely philosophical math teacher saying, ‘Calculus is the language of God’ as a part of one his pearls of wisdom. Being extremely silly, I took that one literally. And when the whole world thought that Roberto Carlos’ free-kick in 97’ was God’s miracle, I dug up on the math. It turned out that God wasn’t being miraculous, he was delivering a Psalm in differential equations.

To set the context for my non-footballing readers of this essay. Roberto Carlos was a Brazilian footballer, who scored one of the world’s greatest freekicks in 1997 against France. So great was it, that the ballboy sitting about 30 feet from the goalpost ducked within the second half of the ball’s trajectory, saving himself from what he had thought would be a punch on his face. But strangely, the ball ended up curving so sharply that it entered the goal.

In 2010, a team of physicists from the École Polytechnique in Paris published a paper in the New Journal of Physics proving it was not a miracle. It was the inevitable consequence of a differential equation, one that governs every single spinning ball in the universe.

This essay traces that equation from its origins in Newton's Second Law, through the forces that shape a football's trajectory, to precisely model why Carlos' free-kick swerved so uncontrollably.

2. Differential equations & Newton’s Second Law

The natural starting point to any trajectory problem is Newton’s 2nd Law, and by modelling it in the form of an Ordinary Differential equation (ODE), we will be able to explore all the forces that govern the ball’s trajectory from the moment it leaves Carlos’ foot.
We know that:

$F_{total} = ma\,, through\ Newton's\ Second\ law,\ wherein\ a = \frac{dv}{dt}\,; Therefore,\ m\frac{dv}{dt} = F(v, t)$

An ODE is defined as any equation relating an unknown function to its derivatives with respect to a single independent variable. Newton's Second Law fits this definition exactly as it produces a first-order Ordinary Differential equation of the form $\frac{dy}{dx} = f(x, y)$, where velocity is the unknown function and time is the independent variable.

Importantly, herein $v = (v_x, v_y, v_z)$ and the consequence of velocity, v, being a vector split into 3 spatial components will be discussed further under section 4. It is also crucial to understand the fact that Carlos’ freekick trajectory wasn’t a set number to be looked up, rather it was a function to be solved for, and the equation governing it starts here.

3. Breakdown of the forces acting on the ball

The right hand side of our equation $m\frac{dv}{dt} = F(v, t)$, talks about the sum of three individual forces in vector form, by specifying directional importance as well.

The first fundamental force acting on the ball is its own weight, producing a gravitational force that points downwards towards the centre of the earth and is represented by this equation.
$F_g = mg\hat{k}\,;\ wherein\ \hat{k}\ is\ a\ unit\ vector\ point\ downwards\ in\ the\ -ve\ z\ direction$

The second, and one of the most important forces acting on the ball that actively offers resistance to the ball's motion to reduce its speed. This is called the Drag force, also referred to as air resistance which ultimately prevents objects from moving infinitely. Since, the drag force acts in the opposite direction of velocity, and velocity itself has components in all three directions, drag spreads across x, y and z directions simultaneously. Therefore, drag is not confined to one axis like gravity, wherever the ball goes, drag opposes it in that direction. It is given by the following equation.
$F_D = \frac{1}{2} \times C_d \times \rho \times A \times |v|v$
Herein, C_d is a dimensionless variable used to quantify an object's resistance to moving through fluids (a substance with a tendency to flow, in this case air is the fluid) and describes how smoothly a body flows through this fluid. ρ is the density of air and A is the cross sectional area of the football. The critical feature that makes the trajectory equation resist analytical solution is the $|v|v$ term, which causes the drag force to grow non-linearly.
Let's take a look at the drag equation in the x direction for instance:

$F_{Dx} = \frac{1}{2} \times C_d \times \rho \times A \times |v|v_x\,,\ and\ we\ know\ that\ |v| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

$Therefore,\ the\ drag\ force\ equation\ becomes\colon\ F_{Dx} = \frac{1}{2} \times C_d \times \rho \times A \times \sqrt{v_x^2 + v_y^2 + v_z^2} \times v_x$

Thus, we can make out that isolation of variable through various differential techniques is difficult as each dimension will have all the velocity vectors (v_x, v_y, v_z). The single term of $|v|v$ in each direction resists the representation as a clean formula, and this is the exact reason why, Spoiler Alert, numerical approaches will be used later on.

The third force acting on the ball and also responsible for the curving nature is called the Magnus force. When the ball moves through the air, on one side of the ball, the incoming airflow creates pressure differences on both sides of the ball, which results in a net rotational force/torque acting on the ball, causing it to rotate.

$F_M = S \times (\omega \times v)$
Herein, S is the empirical surface coefficient, which is known for a football of given size and shape. 'ω', is the angular velocity of the ball, in simple terms it's the change in 'θ' in a given amount of time. This vector quantity is multiplied by the translational/ linear velocity vector which is 'v'.
The product of these two vectors is the mathematical heart of the ball's curvature. Given two vectors, the cross product produces a new vector that is perpendicular to both of these vectors. Since vector, v, points in the x direction and ω spins about the vertical axis, its direction is given as the z axis and therefore the ball will board a trajectory that changes in the y direction, or sideways, causing the ball to curve laterally across the goal. It is also important to note that as the ball curves and decelerates, v,

starts to rotate and therefore the trajectory of the vector is no longer purely in the x direction. This causes the direction of the magnus force to change continuously, and is the exact reason why the ball curves and does not deflect at fixed angles.

4. A coupled, non-linear system that resists solution

As explained above in the drag force section, each directional force produces an equation that is dependent on other directions. This is because of the drag force consisting of a $|v| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ and the Magnus force containing the $(v \times \omega)$, which mixes all three velocity components together. To solve for v_x you would need to already know v_y & v_z, but solving for those requires knowing v_x first. This circular dependency is unbreakable by any analytical method, and it is the precise mathematical reason why the system cannot be solved in closed form.

Now onwards, the essay will be split into two parts, one explaining the bit that can handle an analytical solution and the other for which numerical methods will be used.

5. Spin Decay

While translational trajectory resists analytical solution, the rate of rotation of the ball over time does not. As the ball rotates over time, like translational trajectories, the ball experiences air resistance in the form of aerodynamic frictional torque. This torque is proportional to the spin rate: faster the rate of spin, means greater the resistance to that rotation and faster the spin is reduced. This yields an ODE of the form:

$\frac{d\omega}{dt} = -\alpha \times \omega$

After Variable Separation: $\frac{d\omega}{\omega} = -\alpha\, dt$

Integrating both sides: $\int \frac{d\omega}{\omega} = \int -\alpha\, dt$

Gives the following: $ln\, \omega = -\alpha t + c$

After exponentiating and applying initial condition $\omega(0) = \omega_0$:

$\omega(t) = \omega_0 e^{-\alpha t}$. Using video analysis, it was clear that ω_0= 88 rad/s. And therefore, the equation becomes: $\omega(t) = 88\, e^{-\alpha t}$

This is the essay's only analytical result, which could be derived from variable separation, giving us the spin rate at any point in time of the trajectory, which is a crucial property when modelling the overall trajectory of the ball.

6. The Spinning Spiral - Why was the ball's trajectory impossible to read?

To understand why the ball swerve so uncontrollably, we need to understand Kappa, K, which is a quantification of the curvature of a path, and is given by:

$K\ is\ directly\ proportional\ to\ the\ ratio\ of\ the\ angular\ velocity\ to\ the\ linear\ velocity.$

$K = \frac{\omega}{|v|}$

Now we will consider the ball's complete trajectory as it leaves Carlos' net and moves towards the french goal. Initially, the term $|v|$ falls rapidly as it faces quadratic drag, and therefore v falls quickly and then slowly as the drag force reduces, with deceleration. Alternatively, the rate of change of ω is smaller due to weak aerodynamic torque. Carlos's ball leaves his foot at roughly 34 ms^{-1} but loses nearly a third of its translational speed within the first 10 metres due to quadratic drag. Its spin, however, decays far more slowly. It starts at 88 rad/s and governed by $\omega(t) = 88\, e^{-\alpha t}$, it retains most of its rotation across the full 35 metre flight. Therefore, the ratio or the value of K, grows continuously through the trajectory as $|v|$ keeps on getting smaller and smaller. This describes why initially, the ball is directionless, and once after facing immense quadratic drag, the ball curls sharply into the net.

In summary, this is the spinning spiral decoded by French scientists in their 2010 paper. The ball doesn't form a uniform circular arc that many casual observers would expect but it forms a trajectory of increasing curvature that bends gently at first and then dramatically at the end.

It was also found by these scientists that this great deal of curvature was only accompanied at great distances. For an ordinary Beckham free-kick that is just 20 meters from the goal, you wouldn't expect such a dramatic curve, because the ball simply reaches the net before the value of 'K' grows dramatically.

7. Euler's Method: Reproducing the goal through a Python simulation

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The Coupled ODE established earlier has no analytic solution and therefore numerical methods, like Euler's Method, are used to approximate the trajectory. Euler's method rests on the assumption that if you know the ball's velocity and position at that time, you can use constant forces at that instant to approximate the the velocity and the position a small time, Δt, later.

Systematically, for each time interval n, forces are computed using the data like the velocity and spin rate. Through this acceleration can be computed and then the velocity and position is projected forward, through this equation.

$$v_{n+1} = v_n + \frac{F(v_n)}{m}\Delta t \quad \& \quad x_{n+1} = x_n + v_n \Delta t$$

With Δt = 0.01 seconds and Carlos' initial conditions - $v_0 \approx 34\ ms^{-1}$, spin $\omega_0 = 88$ rad/s, directed to the right of goal. This is the first iteration.

After roughly 150 iterations about 1.5 seconds of flight time, we are able to match the trajectory traced by Carlos' kick. The ball drifts slightly in the first half and after which it takes an increasingly sharp turn inside the far post.

A python code was constructed that modelled the flight's trajectory using computational numerical analysis; ie- made with Euler's Method. It is part of the appendix and can be seen as additional supporting evidence that need not be compulsorily looked at.

8. Conclusion:

Roberto Carlos did not solve a differential equation before taking his free kick. He solved it with his foot, over years of practice, unknowingly calibrating the spin rate and the effect of the forces until the trajectory consistently did what a differential equation could predict.

9. Appendix & Bibliography(not included in the word count)

Python simulation — Euler's method

Run with `dt = 0.0005 s` (3,470 timesteps), initial speed 34 ms^{-1}, spin ω_0 = 88 rad/s, launch angle 16°. Parameters: `m = 0.43 kg`, `C_D = 0.25`, `C_L = 0.33`, `ρ = 1.2 kg/m³`. Spin decay: `ω(t) = ω₀ e^(−αt)`, α = 0.08 s^{-1}.

Distance x	Time t (s)	Lateral y (m)	Height z (m)	Speed |v| (ms^{-1})	Spin ω (rad/s)	κ = ω/|v|
0 m kick	0.000	0.000	0.300	34.00	88.00	2.588
5 m	0.165	+0.371	1.618	31.36	86.89	2.771
10 m wall	0.342	+0.604	2.668	28.97	85.71	2.959
15 m	0.528	+0.691	3.413	26.83	84.48	3.148
20 m	0.722	+0.625	3.813	24.96	83.18	3.332
25 m	0.924	+0.399	3.817	23.36	81.81	3.502
30 m	1.135	+0.004	3.368	22.04	80.37	3.647
35 m goal line	1.356	−0.572	2.397	21.02	78.85	3.751

Speed decay (0 → 35 m)
34.00 → 21.02 ms^{-1}
↓ 38.2% drop

Spin decay (0 → 35 m)
88.00 → 78.85 rad/s
↓ 10.4% drop

Curvature κ growth
2.588 → 3.751
× 1.45 increase across flight

Ball position at goal
y = −0.572 m, z = 2.397 m
within goal frame

I've vibe-coded the algorithm of the Euler's method through Claude and have used its computational abilities to create this simulation table.

Works Cited

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