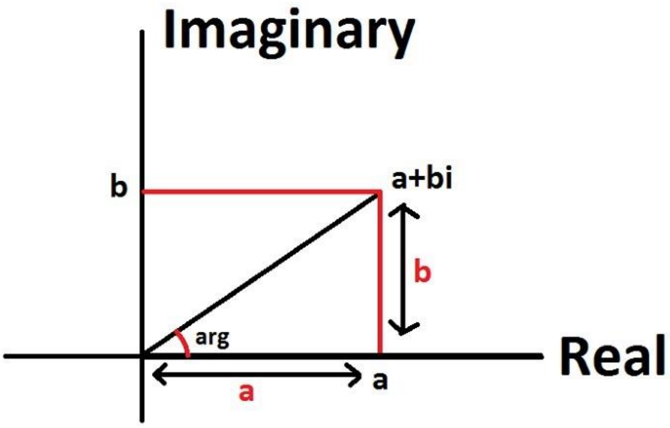
How real are imaginary numbers?

1.1 What are imaginary numbers?

The concept that famous mathematicians had to invent to avoid embarrassment, now underpins your life, from your phone to an MRI scanner. If you were told you had to solve the equation $x^2+1=0$, it may look simple at first, until you actually try to solve it. You will find that you're stuck with $x=\pm\sqrt{-1}$. But how do you find the square root of a negative number? In 1572, Italian mathematician and scientist, Rafael Bombelli, was the first person to set the rules for the multiplication of these numbers, based on the earlier works of Gerolamo Cardano. Then, in the 18th and 19th centuries, famous mathematicians Euler, Gauss and Cauchy were the ones who worked to have this concept widely accepted and formalised. The imaginary number, i, was used to solve this problem. $i=\sqrt{-1}$ and therefore $i^2= -1$. If we go back to our original confusing equation, it now appears to be solvable, $x=\pm i$. With this new information we can solve any quadratic that comes our way. These numbers may seem absolutely useless as they're 'imaginary', but this is far from true.

1.2 What are complex numbers?

We've already met imaginary numbers, but what about their close relative, the complex number. This term is given to any expression in the form $a+bi$, where a is the real part (Re) which is any real number (8, -2, $\sqrt{3}$) and b is the imaginary coefficient (Im), so that bi is any imaginary number (i, $i\sqrt{5}$, $-7i$). These ideas may be hard to imagine in your head, so Argand diagrams can be used to visualise what these numbers look like in the complex plane. An Argand diagram consists of an x and y axis like a regular 2D graph; however, these axes are labelled the real (x) and imaginary (y) parts of a complex number.



These diagrams can be extremely useful when attempting to visualise these numbers as more than just a figment of our imagination. An interesting insight into complex numbers is that, when a complex number (c) is a root of a polynomial, its complex conjugate (c*) is also a root. For example, if 2+5i is one root, 2-5i must be another root of the polynomial. The applications of these numbers are limitless, from electrical engineering to quantum mechanics, wherever you look around you, i will always be there.

2.1 Electrical Engineering

In electrical engineering, the imaginary number may be abstract but is essential. This field of engineering is the study of designs, functions and systems that use electricity. This is often seen as a confusing area of engineering but is necessary for society if we want to maintain our rapidly evolving technological habits. We need these engineers if we want to be able to watch our favourite shows on tv, switch the lights on when we get home or if you want to write an essay about the most fascinating field of maths on a computer. This all revolves around the use of electricity, and perhaps surprisingly the use of the imaginary number.

2.2 AC Circuits

In alternating (AC) circuits, j (the electrical engineering equivalent of i) is seen all the time in complex electricity. Without it, circuit analysis wouldn’t be possible. The voltage (V) and current (I) of a circuit can be expressed in the forms:

$$V_{AC}(t) = |V|\cos(\omega t + \phi_1)$$

And,

$$I_{AC}(t) = |I|\cos\,(\omega t + \phi_2)$$

Where t is the time, ω is the angular frequency, ϕ is the phase of the wave and $|I|$ and $|V|$ are the amplitudes of the waves. Using Euler's identity,

$$e^{j\theta} = \cos(\theta) + \mathrm{j}\sin(\theta).$$

We can express the earlier forms as complex exponentials instead of hard to work with trigonometric values.

$$V(t) = |V|e^{j\phi_1}e^{j\omega t}$$

$$I(t) = |I|e^{j\phi_2}e^{j\omega t}$$

But why do we do this? These formulas are constantly being used in circuit analysis, where methods of calculus are often being used. Having the expressions in the form of exponentials allows for much easier differentiation and integration. Without being in this form, engineers would struggle to do any calculations with the data given, instead they can use the same equations but in a complex exponential form which allows much easier and more efficient use. The use of j allows this form to be expressed; its absence would cause any sort of circuit analysis to be held back significantly. Forcing society to be held back for years, with every calculation being a slow, inefficient process.

2.3 Signal processing

Signal processing is the analysis of a signal in order to extract useful information from it. These signals can be from radios, Wi-Fi or Bluetooth. Whether it be our phone, computer or satellites, these signals from radio waves carry vital information that needs to be decoded for it to be understood and used. This is crucial for most of the devices electrical engineers work with.

But how does the imaginary number relate to this topic? Let me introduce the Fourier equation to you. This equation allows signals to be broken down.

$$f(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\mathrm{d}t$$

What this equation tells us is the magnitude and the phase of the signal. This result is given in the form of a complex number, so j is allowing us to do this entire process. You may also be able to spot the same '$j\omega t$' as seen in the circuit analysis, this demonstrates the significance of j in electrical engineering. Signal processing is one of the most relevant aspects of this field of engineering on a global scale, with applications as personal as allowing you to play games on your phone, to as nationally important as detecting signals hidden in noise in intelligence work. The Fourier transformation is a revolutionary equation that has impacted almost every human on this planet, despite an essential part of it being 'imaginary'.

2.4 Control systems and stability analysis

A control system is a set of devices that manages or regulates the behaviours of other devices. Devices that we use daily such as thermostats, cruise control or even your toaster. These systems are essential in the modern world, they make devices much more efficient, much easier to use and reduce wear significantly.

In electrical engineering, the Laplace transform:

$$s = \sigma + j\omega$$

where s represents a complex variable, σ represents the frequency content of the function, ω represents the phase of the function and σ is the real part and $j\omega$ is the imaginary part; is used to convert a differential equation from a circuit in terms of time, to an algebraic equation in terms of frequency. For engineers, this makes solving circuit analysis problems much easier.

On an Argand diagram, the value on the imaginary axis is $j\omega$ and on the real axis is σ, this can be used with stability analysis for control systems. For a system to be stable, when an additional input is given to the system, it converges in stability. Whereas if it were unstable, the system would diverge in stability, resulting in damaged components and worn devices. On the Argand diagram, if the real axis is negative, the system is stable, if the real axis is positive the system is unstable and will oscillate exponentially. The imaginary axis represents the frequency. The use of a complex number here allows the simultaneous representation of the exponential growth/ decay of the system alongside the oscillation frequency. If a circuit has poles of -5±5j, it will become stable after exposed to an input, because the real part is negative. But if the poles were 5±5j, the system would oscillate in an unstable manner, destroying itself, this could be your thermostat letting your house become a sauna, imagine the cost of that in the winter! Complex numbers here allow us to find out whether you have built a self-destructing system, whilst showing us the amplitude at any given phase. Emphasising the necessity for complex and imaginary numbers in the world we live in.

3.1 Quantum mechanics

Quantum mechanics is often regarded as a petrifying, perplexing area of physics. With all the equations that look as if they were written in another language and the seemingly constant idea that we have no idea how our universe works, it can look intimidating at the best of times. But this terrifying topic has real world applications like you would not believe. All around us, quantum mechanics has tried to explain what happens when light goes through your window, why matter behaves the way it does, and may have even saved your life revolutionising fields of electrical medicine. However, the difference between electrical engineering and quantum mechanics in terms of imaginary numbers is that, in electrical engineering, i is used to make engineers’ lives easier when handling equations, but in quantum mechanics, i is seen naturally built into everything around us.

3.2 The Schrödinger wave equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

The Schrödinger wave equation tells how the quantum state of a system changes over time. This equation has laid the foundation to most of the important chemistry, biology and physics that we rely on today. It allows us to predict how particles behave at both the atomic and sub-atomic scale. This has meant that we can answer questions we were previously confused about, for example how the double slit experiment works. It’s also allowed us to make breakthroughs in various fields, such as quantum computing, which accelerates the processes of regular computers far beyond what was thought to be possible. The Schrödinger equation allows us to do all of this, being the groundbreaking foundation we needed to evolve as a technological society.

But how does this relate to imaginary numbers? Multiplied by the time derivative of the wave function and the reduced Planck's constant, the letter i appears. The presence of i here allows the function to oscillate about the real axis in complex space, instead of growing or decaying exponentially. The use of the complex plane and the multiplication of i results in a rotation of 90°, and not any change in magnitude. It's not a preference or an ideal, it's a necessity for particles in nature. So even though this number is 'imaginary' it's abundant all around us in nature and most definitely is real, or none of us would exist.

4. Conclusion

But now an even bigger question is raised, what does it really mean for something to be real? On the surface you'd say it's something you can hold, measure or see; yet if you take i out of the Schrödinger equation, the building blocks of life fall apart. Perhaps our perception of what is imaginary, or not, isn't as we first thought, maybe if something is essential for the stability of our universe, it deserves to have a more fitting name than 'imaginary'.

'i' was once discovered to bridge a gap in mathematics that confused the best of minds. But now we've noticed that this number has always been around us, we just hadn't noticed it. In electrical engineering, the imaginary number is merely used for convenience, unlike in nature, where i is crucial for life as we know it, from the way particles behave to every Bluetooth or Wi-Fi connection you've ever received. If this doesn't show that this number is real, I don't know what possibly could. So, if someone ever comes up to you and says that the square root of -1 doesn't exist, you can confidently tell them that the universe says otherwise.